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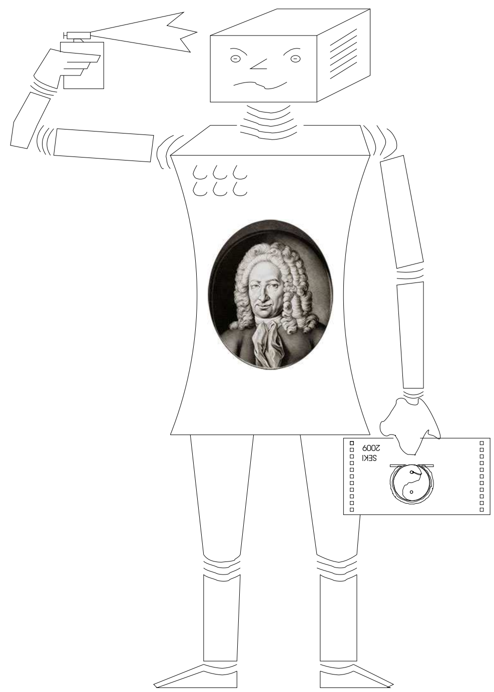

Hilbert's epsilon as an Operator of Indefinite Committed Choice

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Abstract

Paul Bernays and David Hilbert carefully avoided overspecification of Hilbert's ε-operator and axiomatized only what was relevant for their proof-theoretic investigations. Semantically, this left the ε-operator underspecified. In the meanwhile, there have been several suggestions for semantics of the ε as a choice operator. After reviewing the literature on semantics of Hilbert's epsilon operator, we propose a new semantics with the following features: We avoid overspecification (such as right-uniqueness), but admit indefinite choice, committed choice, and classical logics. Moreover, our semantics for the ε supports proof search optimally and is natural in the sense that it does not only mirror some cases of referential interpretation of indefinite articles in natural language, but may also contribute to philosophy of language. Finally, we ask the question whether our ε within our free-variable framework can serve as a paradigm useful in the specification and computation of semantics of discourses in natural language.

Keywords: Hilbert's epsilon Operator, Logical Foundations, Theories of Truth and Validity, Formalized Mathematics, Human-Oriented Interactive Theorem Proving, Automated Theorem Proving, Formal Philosophy of Language, Computational Linguistics

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1 Motivation, Requirements Specification, and Overview

In [Wirth, 2004] we have analyzed the combination of mathematical induction in the liberal style of Fermat's *descente infinie* with state-of-the-art logical deduction into a formal system in which a working mathematician can straightforwardly develop his proofs supported by powerful automation. We have found only a single semantical justification meeting the requirements resulting from this analysis. The means for this semantical justification include a novel semantics for Hilbert's ε-symbol, namely an indefinite choice mirroring some cases of referential interpretation of indefinite articles in natural languages.

Hilbert's ε-symbol is a binder that forms terms; just like Peano's ι-symbol, which is sometimes[1] attributed to Russell and written as $\bar{\iota}$ or as an inverted ι. Roughly speaking, the term $\varepsilon x. A$ formed from a variable x and a formula A denotes *an* object that is *chosen* such that —if possible— A (seen as a predicate on x) holds for it.

For the usefulness of *descriptive terms* such as $\varepsilon x. A$ and $\iota x. A$, we consider the requirements listed below to be the most important ones. Our new indefinite ε-operator satisfies these requirements and—as it is defined by novel semantical techniques—may serve as the paradigm for the design of similar operators satisfying these requirements. As such descriptive terms are of universal interest and applicability, we suppose that our novel treatment will turn out to be useful in many additional areas where logic is designed or applied as a tool for description and reasoning.

Requirement I (Syntax): The syntax must clearly express where exactly a *commitment* to a choice of a special object is required, and where—to the contrary—different objects corresponding with the description may be chosen for different occurrences of the same descriptive term.

Requirement II (Reasoning): In a reductive proof step, it must be possible to replace a descriptive term with a term that corresponds with its description. The soundness of such a replacement must be expressible and should be verifiable in the original calculus.

Requirement III (Semantics): The semantics should be simple, straightforward, natural, formal, and model-based. Overspecification should be avoided carefully. Furthermore, the semantics should be modular and abstract in the sense that it adds the operator to a variety of logics, independently of the details of a concrete logic.

This paper organizes as follows: After a general introduction to the ε in § 2 and a review of the literature on the ε's semantics w.r.t. adequacy and Hilbert's intentions in § 3, we explain and formalize our novel approach to the ε's semantics, first informally in § 4 and then formally in § 5. Finally, in § 6, we discuss some possible implications on philosophy of language, put our ε to test with a list of linguistic examples, and ask the question whether our ε within our free-variable framework can serve as a paradigm useful in the specification and computation of semantics of discourses in natural language.

2 General Introduction to Hilbert's ε

To make this paper accessible to a broader readership, in this § 2, we motivate the ε by introducing first the ι (§ 2.1), then the ε itself (§ 2.2), its proof-theoretic origin (§ 2.3), and our contrasting semantical objective in this paper (§ 2.4) with its emphasis on *definite choice* (§ 2.5) and *committed choice* (§ 2.6). Although the well-informed expert is likely to be amused, he may well skip this § 2 and continue with § 3.

2.1 From the ι to the ε

2.1.1 Intuition behind the ι-Operator

It has turned out not to be completely superfluous to remark that I do not want to hurt any religious feelings with the following example. The delicate subject is chosen for its mnemonic value.

Example 2.1 (ι-binder) *(Buggy!)*

For an informal introduction to the ι-binder, consider Father to be a predicate for which

$$\mathsf{Father}(\mathsf{Heinrich\,III}, \mathsf{Heinrich\,IV})$$

holds, i.e. "Heinrich III is father of Heinrich IV".
Now, "*the* father of Heinrich IV" can be denoted by $\iota x.\, \mathsf{Father}(x, \mathsf{Heinrich\,IV})$, and because this is nobody but Heinrich III, i.e.

$$\mathsf{Heinrich\,III} \;=\; \iota x.\, \mathsf{Father}(x, \mathsf{Heinrich\,IV}),$$

we know that

$$\mathsf{Father}(\iota x.\, \mathsf{Father}(x, \mathsf{Heinrich\,IV}), \mathsf{Heinrich\,IV})$$

Similarly,

$$\mathsf{Father}(\iota x.\, \mathsf{Father}(x, \mathsf{Adam}), \mathsf{Adam}), \tag{2.1.1}$$

and thus $\exists y.\, \mathsf{Father}(y, \mathsf{Adam})$, but, oops! Adam and Eve do not have any fathers.

If you do not agree, you would probably appreciate the following problem that occurs when somebody has God as an additional father.

$$\mathsf{Father}(\mathsf{Holy\ Ghost}, \mathsf{Jesus}) \quad\wedge\quad \mathsf{Father}(\mathsf{Joseph}, \mathsf{Jesus}). \tag{2.1.2}$$

Then the Holy Ghost is *the* father of Jesus and Joseph is *the* father of Jesus, i.e.

$$\mathsf{Holy\ Ghost} = \iota x.\, \mathsf{Father}(x, \mathsf{Jesus}) \;\wedge\; \mathsf{Joseph} = \iota x.\, \mathsf{Father}(x, \mathsf{Jesus}) \tag{2.1.3}$$

which implies something *the* Pope may not accept, namely that

$$\mathsf{Holy\ Ghost} \;=\; \mathsf{Joseph},$$

and he anathematized Heinrich IV in the year 1076:

$$\mathsf{Anathematized}(\iota x.\, \mathsf{Pope}(x), \mathsf{Heinrich\,IV}, 1076). \tag{2.1.4}$$

2.1.2 Semantics of the ι-Operator

There are basically[2] three ways of giving semantics to the ι-terms:

Russell's ι-operator: In [Whitehead & Russell, 1910–1913], the ι-terms do not refer to an object but make sense only in the context of a sentence. This was nicely described already in [Russell, 1905a], without using any symbol for the ι, however; cf. our § 6.2.

Hilbert's ι-operator: To overcome the complex difficulties of that non-referential definition, in [Hilbert & Bernays, 1968/70, Vol. I, p. 392ff.], a completed proof of $\exists!x.\,A$ was required to precede any formation of the term $\iota x.\,A$, which otherwise was not to be considered a well-formed term at all.

Peano's ι-operator: Since the inflexible treatment of Hilbert's ι-operator makes the ι quite impractical and the formal syntax of logic undecidable in general, in Vol. II of the same book, the ε, however, is already given a more flexible treatment. There, the simple idea is to leave the ε-terms uninterpreted, as will be described below. In this paper, we present this more flexible view also for the ι, just as required by an anonymous referee of a previous version of this paper. Moreover, this view is already Peano's original one, cf. ($\bar{\iota}_0$) of Note 1.

At least in non-modal classical logics, it is a well justified standard that *any term denotes*. More precisely—in each model or structure $\mathcal{S}$ under consideration—any occurrence of a proper term must denote an object in the universe of $\mathcal{S}$. (This does not mean that this object has to satisfy properties of ontological existence or definedness in $\mathcal{S}$, cf. § 6.2.4.) Following that standard, to be able to write down $\iota x.\,A$ without further consideration, we have to treat $\iota x.\,A$ as an uninterpreted term about which we only know

$$\exists!x.\,A \quad\Rightarrow\quad A\{x\mapsto \iota x.\,A\} \tag{ι_0}$$

or in different notation

$$(\exists!x.\,(A(x))) \quad\Rightarrow\quad A(\iota x.\,(A(x)))$$

or in set notation

$$\exists!x.\ (x\in A) \quad\Rightarrow\quad \iota x.\ (x\in A)\in A$$

where, for some new y, we can define

$$\exists!x.\,A \quad:=\quad \exists y.\,\forall x.\,\big(\ x{=}y \ \Leftrightarrow\ A\ \big)$$

With (ι_0) as the only axiom for the ι, the term $\iota x.\,A$ has to satisfy A (seen as a predicate on x) only if there exists a unique object such that A holds for it. Moreover, the problems presented in Example 2.1 do not appear because (2.1.1) and (2.1.3) are not valid. Indeed, the description of (2.1.1) lacks existence and the descriptions of (2.1.3) and (2.1.4) lack uniqueness. The price we have to pay here is that—roughly speaking— $\iota x.\,A$ is of no use unless the unique existence $\exists!x.\,A$ can be derived.

2.1.3 Why ε is more useful than ι

Compared to the ι, the ε is more useful because—instead of (ι_0)—it comes with the stronger axiom

$$\exists x.\, A \;\Rightarrow\; A\{x \mapsto \varepsilon x.\, A\} \tag{ε_0}$$

More precisely, as the formula $\exists x.\, A$ (which has to be true to guarantee a meaningful interpretation of the ε-term $\varepsilon x.\, A$) is weaker than the corresponding formula $\exists! x.\, A$ (for the resp. ι-term), the area of useful application is wider for the ε- than for the ι-operator. Moreover, in case of $\exists! x.\, A$, the ε-operator picks the same element as the ι-operator, i.e.

$$\exists! x.\, A \;\Rightarrow\; (\; \varepsilon x.\, A \;=\; \iota x.\, A \;)$$

Although the ι is thus somewhat outdated since the appearance of the superior ε, it is still alive: cf. e.g. [Andrews, 2002], [Nipkow &al., 2002]. For example, in [Nipkow &al., 2002], p. 85, for the *definiendum* $\mu x.\, A$ of the μ-operator (which picks the least natural number satisfying a formula) we find the *definiens*

$$\iota x.\, (\; A \;\wedge\; \forall y.\, (A\{x{\mapsto}y\} \Rightarrow x{\preceq}y) \;)$$

for some new variable y, although the logic of Isabelle/HOL (as given in [Nipkow &al., 2002]) contains an ε, and a *definiens* of

$$\varepsilon x.\, (\; A \;\wedge\; \forall y.\, (A\{x{\mapsto}y\} \Rightarrow x{\preceq}y) \;)$$

would imply the relevant axiom

$$\exists x.\, A \quad\Rightarrow\quad A\{x \mapsto \mu x.\, A\} \quad\wedge\quad \forall y.\, (A\{x{\mapsto}y\} \;\;\Rightarrow\;\; \mu x.\, A \preceq y) \tag{μ_0}$$

for any well-founded total quasi-ordering, while the definition via ι requires antisymmetry in addition. Moreover, a special additional uniqueness proof is required for each unfolding of the definition via ι, hopefully realized automatically, however, with a closely integrated Linear Arithmetic, cf. e.g. [Schmidt-Samoa, 2006]. Indeed, the superior definition via ε is found in [Nipkow &al., 2000].

2.2 What is Hilbert's ε?

As the basic methodology of David Hilbert's formal program is to treat all symbols as meaningless, he does not give us any semantics but only the axiom (ε_0).

Although no meaning is required, it furthers the understanding. And therefore, in [Hilbert & Bernays, 1968/70], the fundamental work on the contributions of David Hilbert and his group to the logical foundations of mathematics, Paul Bernays writes:

$\varepsilon x.\, A$ ... „ist ein Ding des Individuënbereichs, und zwar ist dieses Ding gemäß der inhaltlichen Uebersetzung der Formel (ε_0) *ein solches, auf das jenes Prädikat A zutrifft, vorausgesetzt, daß es überhaupt auf ein Ding des Individuënbereichs zutrifft.*“

[Hilbert & Bernays, 1968/70, Vol. II, p.12, modernized orthography]

$\varepsilon x.\, A$ ... “is an object of the universe for which—according to the semantical translation of the formula (ε_0)—*the predicate A holds, provided that A holds for any object of the universe at all.*” (our translation)

Example 2.2 (ε instead of ι, part I) *(continuing Example 2.1)*

Just as for the ι, for the ε we again have

$$\mathsf{Heinrich\ III} \;=\; \varepsilon x.\, \mathsf{Father}(x, \mathsf{Heinrich\ IV})$$

and

$$\mathsf{Father}(\varepsilon x.\, \mathsf{Father}(x, \mathsf{Heinrich\ IV}), \mathsf{Heinrich\ IV}).$$

But, from the contrapositive of (ε_0) and

$$\neg\mathsf{Father}(\varepsilon x.\, \mathsf{Father}(x, \mathsf{Adam}), \mathsf{Adam}),$$

we now can conclude that

$$\neg\exists y.\, \mathsf{Father}(y, \mathsf{Adam}).$$

2.3 On the ε's Proof-Theoretic Origin

David Hilbert did not need any semantics or precise intention for the ε-symbol because it was introduced merely as a formal syntactical device to facilitate proof-theoretic investigations, motivated by the possibility to get rid of the existential and universal quantifiers via

$$\exists x.\, A \quad\Leftrightarrow\quad A\{x \mapsto \varepsilon x.\, A\} \qquad (\varepsilon_1)$$

and

$$\forall x.\, A \quad\Leftrightarrow\quad A\{x \mapsto \varepsilon x.\, \neg A\} \qquad (\varepsilon_2)$$

Note that ε_0, ε_1, and ε_2 are no ordinal numbers but simply the original labels from [Hilbert & Bernays, 1968/70]. ε_5 is from [Hermes, 1965]. The other labels we use are mostly from [Leisenring, 1969], such as (E2) and (Q2). We recommend [Leisenring, 1969] as an excellent treatment of the subject of the first-order ε-calculus, using a language more modern than the one of [Hilbert & Bernays, 1968/70].

When we remove all quantifiers in a derivation of the Hilbert-style predicate calculus of [Hilbert & Bernays, 1968/70] along (ε_1) and (ε_2), the following transformations occur: Tautologies are turned into tautologies, the axiom schemes[3] $A\{x\mapsto t\} \;\Rightarrow\; \exists x.\, A$ and $\forall x.\, A \;\Rightarrow\; A\{x\mapsto t\}$ are turned into

$$A\{x\mapsto t\} \quad\Rightarrow\quad A\{x \mapsto \varepsilon x.\, A\} \qquad (\varepsilon\textit{-formula})$$

and—roughly speaking w.r.t. two-valued logics—its contrapositive, respectively. The inference steps are turned into inference steps: *modus ponens* into *modus ponens*; instantiation of free variables as well as quantifier introduction into instantiation including ε-terms. Finally, the ε-formula is taken as a new axiom scheme instead of (ε_0) because it has the advantage of being free of quantifiers.

This argumentation is actually the start of the proof transformation of the *1st ε-theorem*, in which the elimination of the ε-formulas did not come easy to Wilhelm Ackermann and David Hilbert.

Theorem 2.3 (Extd. 1st ε-Theorem, [Hilbert & Bernays, 1968/70, Vol. II, p.79f.]) *If we can derive $\exists x_1.\ldots\exists x_r.\, A$ (containing no bound variables besides the ones bound by the prenex $\exists x_1.\ldots\exists x_r.$) from the formulas $P_1,\ldots,P_k$ (containing no bound variables) in the predicate calculus (incl., as axiom schemes, ε-formula and, for equality, reflexivity and substitutability), then, from $P_1,\ldots,P_k$, in the elementary calculus (i.e. tautologies plus modus ponens and instantiation of free variables), we can derive a (finite) disjunction of the form $\bigvee_{i=0}^{s} A\{x_1,\ldots,x_r \mapsto t_{i,1},\ldots,t_{i,r}\}$ in a derivation where bound variables do not occur at all.*

Note that r, s range over natural numbers including 0, and that A, $t_{i,j}$, and P_i are ε-free because otherwise they would have to include (additional) bound variables.

Moreover, the *2nd ε-Theorem* in [Hilbert & Bernays, 1968/70, Vol. II], states that the ε (just as the ι, cf. [Hilbert & Bernays, 1968/70, Vol. I]) is a conservative extension of the predicate calculus in the sense that any formal proof of an ε-free formula can be transformed into a formal proof that does not use the ε at all. Generally, however, it is not a conservative extension to add the ε either with (ε_0), with (ε_1), or with the ε-formula to other first-order logics—may they be weaker such as intuitionistic logic,[4] or stronger such as set theories with axiom schemes over arbitrary terms including the ε, cf. § 3.1.3. Moreover, even in standard first-order logic there is no translation from the formulas containing the ε to formulas not containing it.

2.4 Our Objective

While the historical and technical research on the ε-theorems is still going on and the method of ε-elimination and ε-substitution did not die with Hilbert's programme, this is not our subject here. We are less interested in Hilbert's programme and the consistency of mathematics than in the powerful use of logic in creative processes. And, instead of the tedious syntactical proof transformations, which easily lose their usefulness and elegance within their technical complexity and which—more importantly—can only refer to

an already existing logic, we look for *semantical* means for finding new logics and new applications. And the question that still has to be answered in this field is: *What would be a proper semantics for Hilbert's ε?*

2.5 Indefinite Choice

Just as the ι-symbol is usually taken to be the referential interpretation of the *definite* articles in natural languages, it is our opinion that the ε-symbol should be that of the *indefinite* determiners (articles and pronouns) such as "a(n)" or "some".

Example 2.4 (ε instead of ι, part II) *(continuing Example 2.1)*

It may well be the case that

$$\mathsf{Holy\ Ghost} \;=\; \varepsilon x.\,\mathsf{Father}(x, \mathsf{Jesus}) \quad\wedge\quad \mathsf{Joseph} \;=\; \varepsilon x.\,\mathsf{Father}(x, \mathsf{Jesus})$$

i.e. that "The Holy Ghost is *a* father of Jesus and Joseph is *a* father of Jesus." But this does not bring us into trouble with the Pope because we do not know whether all fathers of Jesus are equal. This will become clearer when we reconsider this example in Example 4.8.

Philosophy of language will be further discussed in § 6.

2.6 Committed Choice

Closely connected to indefinite choice (also called "indeterminism" or "don't care nondeterminism") is the notion of "*committed choice*". For example, when we have a new telephone, we typically *don't care* which number we get, but once the provider has chosen a number for our telephone, we want them to *commit to this choice*, i.e. not to change our phone number between two incoming calls.

Example 2.5 (Committed Choice) *(Buggy!)*

Suppose we want to prove	$\exists x.\,(x \neq x)$
According to (ε_1) from § 2.3 this reduces to	$\varepsilon x.\,(x \neq x) \neq \varepsilon x.\,(x \neq x)$
Since there is no solution to $x \neq x$ we can replace	
$\varepsilon x.\,(x \neq x)$ with anything. Thus, the above reduces to	$\mathsf{0} \neq \varepsilon x.\,(x \neq x)$
and then, by exactly the same argumentation, to	$\mathsf{0} \neq \mathsf{1}$
which is valid.	

Thus we have proved our original formula $\exists x.\,(x \neq x)$, which, however, happens to be invalid. What went wrong? Of course, we have to commit to our choice for all occurrences of the ε-term introduced when eliminating the existential quantifier: If we choose $\mathsf{0}$ on the left-hand side, we have to commit to the choice of $\mathsf{0}$ on the right-hand side, too.

3 Semantics for Hilbert's ε in the Literature

In this § 3, we review the literature on the ε's semantics with a an emphasis on practical adequacy and Hilbert's intentions.

3.1 Right-Unique Semantics

In contrast to the indefiniteness we suggested in § 2.5, in the literature nearly all semantics for Hilbert's ε-operator are functional, i.e. [*right-*] *unique*; cf. [Leisenring, 1969] and the references there.

3.1.1 Ackermann's (II,4) = Bourbaki's (S7) = Leisenring's (E2)

In [Ackermann, 1938] under the label (II,4), in [Bourbaki, 1939ff.] under the label (S7) (where a τ is written for the ε, which must not be confused with Hilbert's τ-operator, cf. Note 4), and in [Leisenring, 1969] under the label (E2), we find the following axiom scheme:

$$\forall x.\ (A_0 \Leftrightarrow A_1) \quad \Rightarrow \quad \varepsilon x.\, A_0 = \varepsilon x.\, A_1 \tag{E2}$$

Contrary to our version (E2′) in Lemma 5.18 of § 5.6, in the standard framework the axiom (E2) imposes a right-unique behavior for the ε-operator, which is based on the extension of the predicate.

Axiom systems including (E2) are called *extensional* because—from a semantical point of view—the value of $\varepsilon x.\, A$ in each semantical structure $\mathcal{S}$ is functionally dependent on the extension of the formula A, i.e. on $\{\ o \mid \mathrm{eval}(\mathcal{S} \uplus \{x \mapsto o\})(A)\ \}$, where 'eval' is the standard evaluation function that maps a structure (or algebra, interpretation) (including a valuation of the free variables) to a function mapping terms and formulas to values.

To get more freedom for the definition of a semantics of the ε, in [Meyer-Viol, 1995] and in [Giese & Ahrendt, 1999] the value of $\varepsilon x.\, A$ may additionally depend on the syntax besides the semantics.[5] It is then given as a function depending on a semantical structure and on the syntactical details of the term $\varepsilon x.\, A$. We read:

> "This definition contains no restriction whatsoever on the valuation of ε-terms."
> [Giese & Ahrendt, 1999, p.177]

This is, however, not true because it imposes the restriction of a right-unique behavior, which denies the possibility of an indefinite behavior, as we will see below.

Note that (E2) has a disastrous effect in intuitionistic logic. This is already the case for its proper consequence $\varepsilon x.\, A_0 \neq \varepsilon x.\, A_1 \ \Rightarrow\ \neg(\forall x.\, A_0 \wedge \forall x.\, A_1)$ which—together with (ε_0) and say "$0 \neq 1$"—turns every classical validity into an intuitionistic one.[6] For the strong consequences of the ε-formula in intuitionistic logic, cf. our Note 4.

3.1.2 Roots of the Right-Uniqueness Requirement

The omnipresence of the right-uniqueness requirement may have its historical justification in the fact that if we expand the dots "…" in the quotation preceding Example 2.2 in § 2.2, the full quotation reads:

> „Das ε-Symbol bildet somit eine Art der Verallgemeinerung des μ-Symbols für einen beliebigen Individuënbereich. Der Form nach stellt es eine Funktion eines variablen Prädikates dar, welches außer demjenigen Argument, auf welches sich die zu dem ε-Symbol gehörige gebundene Variable bezieht, noch freie Variable als Argumente („Parameter") enthalten kann. Der Wert dieser Funktion für ein bestimmtes Prädikat A (bei Festlegung der Parameter) ist ein Ding des Individuënbereichs, und zwar ist dieses Ding gemäß der inhaltlichen Uebersetzung der Formel (ε_0) ein solches, auf das jenes Prädikat A zutrifft, vorausgesetzt, daß es überhaupt auf ein Ding des Individuënbereichs zutrifft."
>
> [Hilbert & Bernays, 1968/70, Vol. II, p.12, modernized orthography]

> "Thus, the ε-symbol forms a kind of generalization of the μ-symbol for arbitrary universes. Syntactically, it provides a function of a variable predicate, which—besides the argument to which the variable bound by the ε-symbol refers—may contain free variables as arguments ("parameters"). The value of this function for a given predicate A (for fixed values of the parameters) is an object of the universe for which—according to the semantical translation of the formula (ε_0)—*the predicate A holds, provided that A holds for any object of the universe at all.*"
>
> (our translation)
>
> ("Syntactically" may be replaced with "Structurally")

Here the word "function" could be understood in its mathematical sense to denote a (right-) unique relation. And, what kind of function could it be but a choice function, choosing an element from the set of objects that satisfy A? Accordingly, at a different place, we read:

> „Darüber hinaus hat das ε die Rolle der Auswahlfunktion, d. h. im Falle, wo $A\,a$ auf mehrere Dinge zutreffen kann, ist εA irgendeines von den Dingen a, auf welche $A\,a$ zutrifft."
>
> [Hilbert, 1928, p. 68]

> "Beyond that, the ε has the rôle of the choice function, i.e. in the case where $A\,a$ may hold for several objects, εA is *an arbitrary one* of the objects a for which $A\,a$ holds."
>
> (our translation)
>
> (in more modern notation, we would possibly write "$A(a)$" for "$A\,a$" and "$\varepsilon x.(A(x))$" for "εA")

3.1.3 Universal and Generalized Choice Functions

Since—in [Hilbert, 1923a, one but last paragraph]—David Hilbert himself seems to have confused the consequences of the ε on the Axiom of Choice (cf. [Rubin & Rubin, 1985], [Howard & Rubin, 1998]), we point out: Although the ε supplies us with a syntactical means for expressing a *universal choice function*, the axioms (E2), (ε_0), (ε_1), and (ε_2) do not imply the Axiom of Choice in set theories, unless the axiom schemes of Replacement (Collection) and Comprehension (Separation, Subset) also range over expressions containing the ε; cf. [Leisenring, 1969], § IV 4.4.

Moreover, to be precise, the notion of a "choice function" must be generalized here because we need a *total* function on the power set of any (non-empty) universe. Thus, a value must be supplied even at the empty set: f is defined to be a *generalized choice function* if $f : \mathrm{dom}(f) \to \bigcup(\mathrm{dom}(f))$ and $\forall x \in \mathrm{dom}(f).\ (x = \emptyset \ \vee\ f(x) \in x)$.

3.1.4 Hans Hermes' (ε_5) and David DeVidi's (vext)

In [Hermes, 1965, p.18], the ε suffers from some overspecification in addition to (E2):

$$\varepsilon x.\,\mathsf{false} \ = \ \varepsilon x.\,\mathsf{true} \tag{ε_5}$$

This sets the value of the generalized choice function f at the empty set to the value of f at the whole universe. For classical logic, we can combine (E2) and (ε_5) into the following axiom of [DeVidi, 1995] for "<u>v</u>ery <u>ext</u>ensional" semantics:

$$\forall x.\left(\begin{array}{cccc} & (\exists y.\,A_0\{x{\mapsto}y\} & \Rightarrow & A_0) \\ \Leftrightarrow & (\exists y.\,A_1\{x{\mapsto}y\} & \Rightarrow & A_1) \end{array}\right) \ \Rightarrow\ \varepsilon x.\,A_0 = \varepsilon x.\,A_1 \tag{vext}$$

Indeed, (vext) implies (E2) and (ε_5). The other direction, however, does not hold for intuitionistic logic, where, roughly speaking, (vext) additionally implies that if the same elements make A_0 and A_1 as true as possible, then the ε-operator picks the same element of this set, even if the suprema $\exists y.\,A_0\{x{\mapsto}y\}$ and $\exists y.\,A_1\{x{\mapsto}y\}$ (in the complete Heyting algebra) are not equally true.

3.1.5 Completeness Aspirations of Leisenring and Asser

Different possible choices for the value of the generalized choice function f at the empty set are discussed in [Leisenring, 1969], but as the consequences of any special choice are quite queer, the only solution that is found to be sufficiently adequate in [Leisenring, 1969] is to consider validity in *any* model given by *each* generalized choice function on the power set of the universe. Notice, however, that even in this case, in each single model, the value of $\varepsilon x.\,A$ is still *functionally* dependent on the extension of A. Roughly speaking, in [Leisenring, 1969] the axioms (ε_1), and (ε_2) from § 2.3 and (E2) from § 3.1.1 are shown to be complete w.r.t. this semantics of the ε in first-order logic.

This completeness makes it unlikely that this semantics exactly matches Hilbert's intentions: Indeed, if Hilbert's intended semantics for the ε could be completely captured by adding the single and straightforward axiom (E2), this axiom would not have been omitted in [Hilbert & Bernays, 1968/70]. It is my opinion that the reason for this omission is that

Hilbert's intentions for the ε were not right-unique but indefinite: If Hilbert had intended a right-unique behavior, it would not be impossible to derive (E2) from his axiomatization!

Completeness—detached from practical usefulness, but the theoreticians' favorite puzzle—has misled others, too: In [Asser, 1957] the objective is to find a semantics such that Hilbert's ε-calculus of [Hilbert & Bernays, 1968/70] is sound and *complete* for it. This semantics, however, has to depend on the details of the syntactical form of the ε-terms and, moreover, turns out to be necessarily so artificial that in [Asser, 1957] the author himself does not recommend it and admits not to believe that Hilbert could have intended it:

> „Allerdings ist dieser Begriff von Auswahlfunktion so kompliziert, daß sich seine Verwendung in der inhaltlichen Mathematik kaum empfiehlt.“ [Asser, 1957, p. 59, modernized orthography]
>
> “This notion of a choice function, however,” (i.e. the type-3 choice function, providing a semantics for the ε-operator) “is so intricate that its application in informal mathematics is hardly to be recommended.” (our translation)
> (“informal mathematics” may be replaced with “intuitive mathematics”, “naïve mathematics”, or “mathematics with semantical contents”)

> „Angesichts der Kompliziertheit des Begriffs der Auswahlfunktion dritter Art ergibt sich die Frage, ob bei Hilbert-Bernays (“ ... „) wirklich beabsichtigt war, diesen Begriff von Auswahlfunktion axiomatisch zu beschreiben. Aus der Darstellung bei Hilbert-Bernays glaube ich entnehmen zu können, daß das nicht der Fall ist,“ [Asser, 1957, p. 65, modernized orthography]
>
> “The intricacy of the notion of the type-3 choice function puts up the question whether the intention in [Hilbert & Bernays, 1968/70] (” ... “) really was to describe this notion axiomatically. I believe I can draw from the presentation in [Hilbert & Bernays, 1968/70] that that is not the case,” (our translation)

3.1.6 My Assumption on Hilbert's Intentions

The statements of Bernays and Hilbert in German language cited in § 3.1.2 are ambiguous with respect to the question of an intended (right-) unique behavior of the ε-operator. Hilbert probably wanted to have what today we call “*committed choice*”, but simply used the word “function” for the following three reasons: Hilbert was not too much interested in semantics anyway. The technical term “committed choice” did not exist at Hilbert's time. Last but not least, right-uniqueness conveniently serves as a global commitment to any choice and thereby avoids the problem illustrated in Example 2.5 of § 2.6.

But the price we would have to pay for such an overspecification is high: Right-Uniqueness restricts operationalization (cf. § 4.6) and applicability: Cf. e.g. [Geurts, 2000] and our § 6.5 for the price of right-uniqueness in capturing the semantics of sentences in natural language.

And what we are going to show in this paper is that there is no reason to pay that price!

3.2 Indefinite Semantics in the Literature

The only occurrences of an indefinite semantics for Hilbert's ε in the literature seem to be [Blass & Gurevich, 2000] and the references there.

Consider the formula $\varepsilon x.\,(x=x) \;=\; \varepsilon x.\,(x=x)$ from [Blass & Gurevich, 2000] or the even simpler

$$\varepsilon x.\,\mathsf{true} \;=\; \varepsilon x.\,\mathsf{true} \tag{Reflex}$$

which may be valid or not, depending on the question whether the same object is taken on both sides of the equation or not. In natural language this like

"Something is equal to something."

whose truth is indefinite. If you do not think so, consider $\varepsilon x.\,\mathsf{true} \neq \varepsilon x.\,\mathsf{true}$ in addition, i.e. "Something is unequal to something.", and notice that the two sentences seem to be contradictory.

In [Blass & Gurevich, 2000], Kleene's strong three-valued logic is taken as a mathematically elegant means to solve the problems with indefiniteness. In spite of the theoretical significance of this solution, however, from a practical point of view, Kleene's strong three-valued logic severely restricts its applicability. In applications, a logic is not an object of investigation but a meta-logical tool, and logical arguments are never made explicit because the presence of logic is either not realized at all or taken to be trivial, even by academics (unless they are formalists), cf. e.g. [Pinkal &al., 2001, p.14f.], for Wizard of Oz studies with young students. Thus, regarding applications, we have to stick to our common meta-logic, which in the western world is a subset of (modal) classical logic. A western court may accept that Lee Harvey Oswald killed John F. Kennedy as well as that he did not; but cannot accept a third possibility, a *tertium*, as required for Kleene's strong three-valued logic, and especially not the interpretation given in [Blass & Gurevich, 2000] that he *both* did and did not kill him, which directly contradicts any common sense.

4 Introduction to Our Novel Indefinite Free-Variable Semantics

4.1 Free γ- and Free δ-Variables

Before we can introduce to our treatment of the ε, we have to provide some technical background. Cf. [Wirth, 2004] for a technically more detailed introduction.

In this § 4.1, we will introduce free γ-, δ^--, and δ^+-variables. Free variables frequently occur in mathematical practice. Their logical function varies locally. It is typically determined implicitly by the context and the obviously intended semantics.

In this paper, however, we make this function explicit by using disjoint sets of variable-symbols for different functions. The classification of a free variable is indicated by adjoining the respective γ, δ^-, or δ^+ to the upper right of the symbol for the variable.

As already noted in [Russell, 1919, p.155], in mathematical practice, the free variables a^{free} and b^{free} in the (quasi-) formula

$$(a^{\mathrm{free}}+b^{\mathrm{free}})^2 \quad = \quad (a^{\mathrm{free}})^2+2\,a^{\mathrm{free}}\,b^{\mathrm{free}}+(b^{\mathrm{free}})^2$$

obviously have a universal intention and the quasi-formula itself is not meant to denote a propositional function but actually stands for the closed formula

$$\forall a,b.\ \big(\ (a+b)^2 \quad = \quad (a)^2+2\,a\,b+(b)^2\ \big)$$

In this paper, however, we indicate by

$$(a^{\delta^-}+b^{\delta^-})^2 \quad = \quad (a^{\delta^-})^2+2\,a^{\delta^-}\,b^{\delta^-}+(b^{\delta^-})^2$$

a proper formula with free δ^--variables, which—independently of its context—is logically equivalent to the universally quantified formula.

Changing from universal to existential intention, it is somehow clear that the linear system

$$\begin{pmatrix}2&3\\5&7\end{pmatrix}\begin{pmatrix}x^{\mathrm{free}}\\y^{\mathrm{free}}\end{pmatrix}=\begin{pmatrix}11\\13\end{pmatrix}$$

asks us to find solutions for x^{free} and y^{free}. We make this intention syntactically explicit by writing

$$\begin{pmatrix}2&3\\5&7\end{pmatrix}\begin{pmatrix}x^{\gamma}\\y^{\gamma}\end{pmatrix}=\begin{pmatrix}11\\13\end{pmatrix}$$

instead. This formula with free γ-variables is not only logically equivalent to

$$\exists x,y.\ \left(\ \begin{pmatrix}2&3\\5&7\end{pmatrix}\begin{pmatrix}x\\y\end{pmatrix}=\begin{pmatrix}11\\13\end{pmatrix}\ \right)$$

but may additionally enable us to retrieve the solutions for x^{γ} and y^{γ} as the substitutions for x^{γ} and y^{γ} chosen in a formal proof.

Finally, the free δ^+-variables are to represent our ε-terms in the end. The names γ, δ^-, and δ^+ refer to the classification of reductive inference rules into α-, β-, γ-, and δ-rules of [Smullyan, 1968], as used in the following § 4.2.

4.2 γ- and δ-Rules

Suppose we want to prove the existential property $\exists x.\, A$. The γ-rules of old-fashioned inference systems (such as [Gentzen, 1935] or [Smullyan, 1968], e.g.) require us to choose a *fixed* witnessing term t as a substitute for the bound variable *immediately* when eliminating the quantifier.

Let A be a formula. We do not permit binding of variables that already occur bound in a term or formula; that is: $\forall x.\ A$ is only a formula if no binder on x already occurs in A. The simple effect is that our formulas are easier to read and our γ- and δ-rules can replace *all* occurrences of x. Moreover, we assume that all binders have minimal scope, e.g. $\forall x, y.\ A \wedge B$ reads $(\forall x.\ \forall y.\ A) \wedge B$. Let Γ and Π be *sequents*, i.e. disjunctive lists of formulas.

γ-rules: Let t be any term:

$$\frac{\Gamma \quad \exists x.A \quad \Pi}{A\{x{\mapsto}t\} \quad \Gamma \quad \exists x.A \quad \Pi} \qquad\qquad \frac{\Gamma \quad \neg\forall x.A \quad \Pi}{\overline{A\{x{\mapsto}t\}} \quad \Gamma \quad \neg\forall x.A \quad \Pi}$$

Note that $\overline{A}$ is the *conjugate* of the formula A, i.e. B if A is of the form $\neg B$, and $\neg A$ otherwise. Moreover, in the good old days when trees grew upwards, Gerhard Gentzen (1909–1945) would have inverted the inference rules such that passing the line means consequence. In our case, passing the line means reduction, and trees grow downwards.

More modern inference systems, however, (such as the ones in [Fitting, 1996]) enable us to delay the crucial choice of the term t until the state of the proof attempt may provide more information to make a successful decision. This delay is achieved by introducing a special kind of variable, called "dummy" in [Prawitz, 1960], "free" in [Fitting, 1996] and in Footnote 11 of [Prawitz, 1960], and "meta" in the field of planning and constraint solving. We call these variables *free γ-variables* and write them like x^γ. When these additional variables are available, we can reduce $\exists x.\, A$ first to $A\{x{\mapsto}x^\gamma\}$ and then sometime later in the proof we may globally substitute x^γ with an appropriate term.

The addition of the free γ-variables changes the notion of a term but not the γ-rules, whereas it becomes visible in the δ-rules. δ-rules introduce free δ-variables. The free δ-variables are also called "parameters" or "eigenvariables" and typically stand for arbitrary objects of which nothing is known. Now the occurrence of such a free δ-variable must be disallowed in the terms that may be substituted for those free γ-variables which have already been in use when an application of a δ-rule introduced this free δ-variable. The reason for this restriction of substitution for free γ-variables is that the dependence or scoping of the quantifiers must somehow be reflected in a dependence of the free variables. This dependence is to be captured in a binary relation on the free variables, called *variable-condition*.

Indeed, it is sometimes unsound to instantiate a free γ-variable x^γ with a term containing a free δ-variable y^δ that was introduced later than x^γ:

Example 4.1 The formula $\exists x.\ \forall y.\ (x\!=\!y)$
is not generally valid. We can start a proof attempt as follows:

γ-step: $\forall y.\ (x^\gamma\!=\!y),\quad \exists x.\ \forall y.\ (x\!=\!y)$

δ-step: $(x^\gamma\!=\!y^\delta),\quad \exists x.\ \forall y.\ (x\!=\!y)$

Now, if the free γ-variable x^γ could be substituted by the free δ-variable y^δ, we would get the tautology $(y^\delta\!=\!y^\delta)$, i.e. we would have proved an invalid formula. To prevent this, the δ-step has to record (x^γ, y^δ) in a variable-condition, where (x^γ, y^δ) means that x^γ is somehow "necessarily older" than y^δ, so that we must not instantiate the free γ-variable x^γ with a term containing the free δ-variable y^δ.

Starting with an empty variable-condition, we extend the variable-condition during a proof by δ-steps and by steps that globally instantiate γ- and δ^+-variables. This kind of instantiation of *rigid* variables is only sound if the resulting variable-condition is still acyclic after adding, for each free variable x^{free} instantiated with a term t and for each free variable z^{free} occurring in t, the pair $(z^{\mathrm{free}}, x^{\mathrm{free}})$ to the variable-condition.

To make things more complicated, there are basically two different versions of the δ-rules: standard δ^--rules (also simply called "δ-rules") and δ^+-rules (also called "*liberalized* δ-rules"). They differ in the kind of free δ-variable they introduce and—crucially—in the way they enlarge the variable-condition, depicted to the lower right of the bar:

δ^--rules: Let x^{δ^-} be a new free δ^--variable:

$$\frac{\varGamma\quad \forall x.A\quad \varPi}{A\{x\mapsto x^{\delta^-}\}\quad \varGamma\quad \varPi}\qquad \mathcal{V}_{\gamma\delta^+}(\varGamma\ \forall x.A\ \varPi)\times\{x^{\delta^-}\}$$

$$\frac{\varGamma\quad \neg\exists x.A\quad \varPi}{\overline{A\{x\mapsto x^{\delta^-}\}}\quad \varGamma\quad \varPi}\qquad \mathcal{V}_{\gamma\delta^+}(\varGamma\ \neg\exists x.A\ \varPi)\times\{x^{\delta^-}\}$$

δ^+-rules: Let x^{δ^+} be a new free δ^+-variable:

$$\frac{\varGamma\quad \forall x.A\quad \varPi}{A\{x\mapsto x^{\delta^+}\}\quad \varGamma\quad \varPi}\qquad \begin{array}{l}\{(x^{\delta^+}, \overline{A\{x\mapsto x^{\delta^+}\}}\,)\}\\ \mathcal{V}_{\mathrm{free}}(\forall x.A)\times\{x^{\delta^+}\}\end{array}$$

$$\frac{\varGamma\quad \neg\exists x.A\quad \varPi}{\overline{A\{x\mapsto x^{\delta^+}\}}\quad \varGamma\quad \varPi}\qquad \begin{array}{l}\{(x^{\delta^+}, A\{x\mapsto x^{\delta^+}\})\}\\ \mathcal{V}_{\mathrm{free}}(\neg\exists x.A)\times\{x^{\delta^+}\}\end{array}$$

Notice that $\mathcal{V}_{\gamma\delta^+}(\varGamma\ \forall x.A\ \varPi)$ denotes the set of the free γ- and δ^+-variables occurring in the whole upper sequent, whereas $\mathcal{V}_{\mathrm{free}}(\forall x.A)$ denotes the set of all free (γ-, δ^--, δ^+-) variables, but only the ones occurring in the *principal formula* $\forall x.A$. The smaller variable-conditions generated by the δ^+-rules mean more proofs. Indeed, the δ^+-rules enable additional proofs on the same level of *multiplicity* (i.e. the number of repeated γ-steps applied to the identical principal formula); cf. e.g. [Wirth, 2004, Example 2.8, p. 21]. For certain classes of

theorems, some of these proofs are exponentially and even non-elementarily shorter than the shortest proofs which apply only δ^--rules; for a survey cf. [Wirth, 2004, § 2.1.5]. Moreover, the δ^+-rules provide additional proofs that are not only shorter but also more natural and easier to find both automatically and for human beings; cf. the discussion on design goals for inference systems in [Wirth, 2004, § 1.2.1], and the proof of the limit theorem for + in [Wirth, 2006]. All in all, the name "liberalized" for the δ^+-rules is indeed justified: They provide more freedom to the prover.[7]

Moreover, note that the singleton sets indicated to the upper right of the bar of the above δ^+-rules are to augment another global binary relation besides the variable-condition, namely a function called the *choice-condition*. This will be explained in § 4.5f.

There is a popular alternative to variable-conditions, namely Skolemization, where the free δ-variables become functions (i.e. their order is incremented) and the δ^-- and δ^+-rules give them the free γ-variables of $\mathcal{V}_\gamma(\Gamma\ \forall x.A\ \Pi)$ and $\mathcal{V}_\gamma(\forall x.A)$, resp., as initial arguments. Then, the occur-check of unification implements the restrictions on substitution of free γ-variables. In some inference systems, however, Skolemization is unsound (e.g. for higher-order systems such as the one in [Kohlhase, 1998] or the system in [Wirth, 2004] for *descente infinie*) or inappropriate (e.g. in the matrix systems of [Wallen, 1990]). We prefer inference systems with variable-conditions as this is a simpler, more general, and not less efficient approach compared to Skolemizing inference systems. Notice that variable-conditions do not add unnecessary complexity: Firstly, if variable-conditions are superfluous we can work with an empty variable-condition as if there would be no variable-condition at all. Secondly, we will need the variable-conditions anyway for our choice-conditions, which again are needed to formalize our novel approach to Hilbert's ε-operator.

4.3 Quantifier Elimination and Subordinate ε-terms

Before we can introduce to our treatment of the ε, we also have to get more acquainted with the ε in general.

The elimination of $\forall$- and $\exists$-quantifiers with the help of ε-terms (cf. § 2.3) may be more difficult than expected when some ε-terms become "subordinate" to others.

Definition 4.2 (Subordinate) An ε-term $\varepsilon v.\, B$ (or, more generally, a binder on v together with its scope B) is *superordinate* to an (occurrence of an) ε-term $\varepsilon x.\, A$ if

1. $\varepsilon x.\, A$ is a subterm of B and
2. an occurrence of the variable v in $\varepsilon x.\, A$ is free in B
   (i.e. the binder on v binds an occurrence of v in $\varepsilon x.\, A$).

An (occurrence of an) ε-term a is *subordinate* to an ε-term $\varepsilon v.\, B$ (or, more generally, a binder on v together with its scope B) if $\varepsilon v.\, B$ is superordinate to a.

In [Hilbert & Bernays, 1968/70, Vol. II, p. 24], these subordinate ε-terms, which are responsible for the difficulty to prove the ε-theorems constructively, are called "untergeordnete ε-Ausdrücke". Note that we do not use a special name for ε-terms with free occurrences of variables—such as "ε-Ausdrücke" ("quasi ε-terms") instead of "ε-Terme" ("ε-terms")—but simply call them "ε-terms", too.

Example 4.3 (Quantifier Elimination and Subordinate ε-Terms)
Consider the formula $\forall x.\ \exists y.\ \forall z.\ \mathsf{P}(x,y,z)$. Let us apply (ε_1) and (ε_2) from § 2.3 to remove the three quantifiers completely. We introduce the following abbreviations:

$$\begin{array}{lcl|lcl}
z_a(x)(y) & = & \varepsilon z.\ \neg\mathsf{P}(x,y,z) & x_a & = & \varepsilon x.\ \neg\mathsf{P}(x, y_a(x), z_a(x)(y_a(x))) \\
 & & & x_b & = & \varepsilon x.\ \neg\mathsf{P}(x, y_a(x), z_a(x)(y_b(x))) \\
y_a(x) & = & \varepsilon y.\ \mathsf{P}(x,y,z_a(x)(y)) & x_c & = & \varepsilon x.\ \neg\mathsf{P}(x, y_b(x), z_a(x)(y_b(x))) \\
y_b(x) & = & \varepsilon y.\ \forall z.\ \mathsf{P}(x,y,z) & x_d & = & \varepsilon x.\ \neg\forall z.\ \mathsf{P}(x, y_b(x), z) \\
 & & & x_e & = & \varepsilon x.\ \neg\exists y.\ \forall z.\ \mathsf{P}(x,y,z)
\end{array}$$

When we eliminate inside–out (i.e. start with the elimination of $\forall z.$) the transformation is
$\forall x.\ \exists y.\ \mathsf{P}(x,y,z_a(x)(y))$, $\quad \forall x.\ \mathsf{P}(x, y_a(x), z_a(x)(y_a(x)))$, $\quad \mathsf{P}(x_a, y_a(x_a), z_a(x_a)(y_a(x_a)))$

When we eliminate outside–in (i.e. start with the elimination of $\forall x.$) the transformation is
$\exists y.\ \forall z.\ \mathsf{P}(x_e,y,z)$, $\quad \forall z.\ \mathsf{P}(x_e, y_b(x_e), z)$, $\quad \mathsf{P}(x_e, y_b(x_e), z_a(x_e)(y_b(x_e)))$,
$\ldots,$ $\quad \mathsf{P}(x_a, y_a(x_a), z_a(x_a)(y_a(x_a)))$
where the dots represent the rewritings of x_e over x_d, x_c, x_b to x_a (four times) and of y_b to y_a (twice in addition).

Note that the resulting formula is the same in both cases. Indeed, it does not depend on the order in which we eliminate the quantifiers. Moreover, notice that this formula is quite deep. Indeed, in general n nested quantifiers result in an ε-nesting depth of 2^n-1 and huge ε-terms (such as x_a) occur up to n times with commitment to their choice. Let us have a closer look to see this. If we write the resulting formula as

$$\mathsf{P}(x_a, y_c, z_d) \tag{4.3.1}$$

by setting $y_c = y_a(x_a)$, and $z_d = z_a(x_a)(y_a(x_a))$, then we have

$$\begin{array}{lcllcll}
z_d & = & \varepsilon z.\ \neg\mathsf{P}(x_a, y_c, z) & & & & (4.3.2) \\
y_c & = & \varepsilon y.\ \mathsf{P}(x_a, y, z_c(y)) & & & & (4.3.3) \\
 & & \text{with} \quad z_c(y) & = & \varepsilon z.\ \neg\mathsf{P}(x_a, y, z) & & (4.3.4) \\
x_a & = & \varepsilon x.\ \neg\mathsf{P}(x, y_a(x), z_b(x)) & & & & (4.3.5) \\
 & & \text{with} \quad z_b(x) & = & \varepsilon z.\ \neg\mathsf{P}(x, y_a(x), z) & & (4.3.6) \\
 & & \text{and} \quad y_a(x) & = & \varepsilon y.\ \mathsf{P}(x, y, z_a(x)(y)) & & (4.3.7) \\
 & & & & \text{with} \quad z_a(x)(y) & = \ \varepsilon z.\ \neg\mathsf{P}(x,y,z) & (4.3.8)
\end{array}$$

Firstly, note that the free variables x and y in the ε-terms $z_c(y)$, $z_b(x)$, $y_a(x)$, $z_a(x)(y)$ are actually bound by the next ε to the left, to which the respective ε-terms thus become subordinate. For example, the ε-term $z_c(y)$ is subordinate to the ε-term y_c. Secondly, the top ε-binders on the right-hand sides of the defining equations are exactly those that require a commitment to their choice. This means that each of z_a, z_b, z_c, z_d and each of y_a, y_c may be chosen differently without affecting soundness of the equivalence transformation. Note that the variables are strictly nested into each other. Thus we must choose in the order of z_a, y_a, z_b, x_a, z_c, y_c, z_d. Moreover, for z_c, z_b, y_a, z_a we actually have to choose a function instead of a simple value. In Hilbert's view, however, there are neither functions nor objects at all, but only terms, where x_a reads

$$\varepsilon x.\ \neg\mathsf{P}\left(\begin{array}{l} x, \\ \varepsilon y_\alpha.\ \mathsf{P}(x, y_\alpha, \varepsilon z_\alpha.\ \neg\mathsf{P}(x, y_\alpha, z_\alpha)), \\ \varepsilon z_\beta.\ \neg\mathsf{P}(\ x,\ \ \varepsilon y_\alpha.\ \mathsf{P}(x, y_\alpha, \varepsilon z_\alpha.\ \neg\mathsf{P}(x, y_\alpha, z_\alpha)),\ \ z_\beta\ ), \end{array}\right)$$

and y_c and z_d take several lines more to write them down.

For $\forall x.\ \forall y.\ \forall z.\ \mathsf{P}(x,y,z)$ instead of $\forall x.\ \exists y.\ \forall z.\ \mathsf{P}(x,y,z)$, we get the same exponential growth of nesting depth as in Example 4.3 above, when we completely eliminate the quantifiers using (ε_2). The only difference is that we get additional occurrences of '$\neg$' in y_a, y_b, and y_c. But when we have quantifiers of the same kind like '$\exists$' or '$\forall$', we had better choose them in parallel, e.g., for $\forall x.\ \forall y.\ \forall z.\ \mathsf{P}(x,y,z)$ we choose $v_a := \varepsilon v.\ \neg\mathsf{P}(1^{\mathrm{st}}(v), 2^{\mathrm{nd}}(v), 3^{\mathrm{rd}}(v))$, and then take $\mathsf{P}(1^{\mathrm{st}}(v_a), 2^{\mathrm{nd}}(v_a), 3^{\mathrm{rd}}(v_a))$ as result of the elimination.

Roughly speaking, in today's theorem proving, cf. e.g. [Fitting, 1996], [Wirth, 2004], the exponential explosion of term depth of Example 4.3 is avoided by an outside–in removal of δ-quantifiers *without removing the quantifiers below ε-binders* and by a replacement of γ-quantified variables with free γ-variables. For the case of Example 4.3, this yields $\mathsf{P}(x_e, y^\gamma, z_e)$ with $z_e = \varepsilon z.\ \neg\mathsf{P}(x_e, y^\gamma, z)$ and $x_e = \varepsilon x.\ \neg\exists y.\ \forall z.\ \mathsf{P}(x,y,z)$. Thus, in general, the nesting of binders for the complete elimination of a prenex of n quantifiers does not become deeper than $\frac{1}{4}(n{+}1)^2$.

Moreover, if we are only interested in reduction and not in equivalence transformation of a formula, we can abstract Skolem terms from the ε-terms and just reduce to the formula $\mathsf{P}(x^\delta, y^\gamma, z^\delta(y^\gamma))$. In a non-Skolemizing inference system with a variable-condition we get $\mathsf{P}(x^\delta, y^\gamma, z^\delta)$ instead, with $\{(y^\gamma, z^\delta)\}$ as an extension to the variable-condition. Note that with Skolemization or variable-conditions we have no growth of nesting depth at all, and the same will be the case for our approach to ε-terms.

4.4 Do not be afraid of Indefiniteness!

From the discussion in § 2.5 and § 3, one could get the impression that an indefinite logical treatment of the ε is not easy to find. Indeed, on the first sight, there is the problem that some standard axiom schemes cannot be taken for granted, such as substitutability

$$s = t \quad \Rightarrow \quad f(s) = f(t)$$

(note that this is similar to (E2) of § 3.1.1 when we take logical equivalence as equality!) and such as reflexivity

$$t = t$$

(note that (Reflex) of § 3.2 is an instance of this!)

This means that it is not definitely okay to replace a subterm with an equal term and that even syntactically equal terms may not be definitely equal.

It may be interesting to see that—in computer programs—we are quite used to committed choice and to an indefinite behavior of choosing, and that the violation of substitutability and even reflexivity is no problem there:

Example 4.4 (Violation of Substitutability and Reflexivity in Programs)

In the implementation of the specification of the web-based hypertext system of [Mattick & Wirth, 1999] we needed a function that chooses an element from a set implemented as a list. Its ML code is

```
fun choose s = case s of Set (i :: _) => i | _ => raise Empty;
```

And, of course, it simply returns the first element of the list. For another set that is equal—but where the list may have another order—the result may be different. Thus, the

behavior of the function `choose` is indefinite for a given set, but any time it is called for an implemented set, it chooses a special element and *commits to this choice*, i.e. when called again, it returns the same value. In this case we have `choose s` $=$ `choose s`, but `s` $=$ `t` does not imply `choose s` $=$ `choose t`. In an implementation where some parallel reordering of lists may take place, even `choose s` $=$ `choose s` may be wrong.

From this example we may learn that the question of `choose s` $=$ `choose s` may be indefinite until the choice steps have actually been performed. *This is exactly how we will treat our* ε. The steps that are performed in logic are proof steps.

Thus, on the one hand, when we want to prove

$$\varepsilon x.\,\mathsf{true} \;=\; \varepsilon x.\,\mathsf{true}$$

we can choose $\mathsf{0}$ for both occurrences of $\varepsilon x.\,\mathsf{true}$, get $\mathsf{0}=\mathsf{0}$, and the proof is successful. On the other hand, when we want to prove

$$\varepsilon x.\,\mathsf{true} \;\neq\; \varepsilon x.\,\mathsf{true}$$

we can choose $\mathsf{0}$ for one occurrence and $\mathsf{1}$ for the other, get $\mathsf{0}\neq\mathsf{1}$, and the proof is successful again. This procedure may seem wondrous again, but is very similar to something quite common with free γ-variables, cf. § 4.1: On the one hand, when we want to prove

$$x^{\gamma} = y^{\gamma}$$

we can choose $\mathsf{0}$ to substitute for both x^{γ} and y^{γ}, get $\mathsf{0}=\mathsf{0}$, and the proof is successful. On the other hand, when we want to prove

$$x^{\gamma} \neq y^{\gamma}$$

we can choose $\mathsf{0}$ to substitute for x^{γ} and $\mathsf{1}$ to substitute for y^{γ}, get $\mathsf{0}\neq\mathsf{1}$, and the proof is successful again.

4.5 Replacing ε-terms with Free δ^+-Variables

There is an important difference between the inequations $\varepsilon x.\,\mathsf{true} \neq \varepsilon x.\,\mathsf{true}$ and $x^{\gamma} \neq y^{\gamma}$ at the end of the previous § 4.4: The latter does not violate the reflexivity axiom! And we are going to cure the violation of the former immediately with the help of a special kind of free variables, namely our *free δ^+-variables*, cf. § 4.1. Now, instead of $\varepsilon x.\,\mathsf{true} \neq \varepsilon x.\,\mathsf{true}$ we write $x^{\delta^+} \neq y^{\delta^+}$ and remember what these free δ^+-variables stand for by storing this into a function C, called a *choice-condition*:

$$\begin{aligned} C(x^{\delta^+}) &:= \mathsf{true}, \\ C(y^{\delta^+}) &:= \mathsf{true}. \end{aligned}$$

For a first step, suppose that our ε-terms are not subordinate to any outside binder, cf. Definition 4.2. Then, we can replace an ε-term $\varepsilon z.\,A$ with a new free δ^+-variable z^{δ^+} and extend the partial function C by

$$C(z^{\delta^+}) \;:=\; A\{z \mapsto z^{\delta^+}\}.$$

By this procedure we can eliminate all ε-terms without loosing any syntactical information.

As a first consequence of this elimination, the substitutability and reflexivity axioms are immediately regained, and the problems discussed in § 4.4 disappear.

A second reason for replacing the ε-terms with free δ^+-variables is that the latter can solve the question whether a committed choice is required: We can express—on the one hand—a committed choice by using a single free δ^+-variable and—on the other hand—a choice without commitment by using several variables with the same choice-condition.

Indeed, this also solves our problems with committed choice of Example 2.5 of § 2.6: Now, again using (ε_1), $\exists x.\,(x \neq x)$ reduces to $x^{\delta^+} \neq x^{\delta^+}$ with

$$C(x^{\delta^+}) \quad := \quad (x^{\delta^+} \neq x^{\delta^+})$$

and the proof attempt immediately fails due to the now regained reflexivity axiom.

As the second step, we still have to explain what to do with subordinate ε-terms. If the ε-term $\varepsilon z.\,A$ contains free occurrences of exactly the distinct variables $v_0, \ldots, v_{l-1}$, then we have to replace this ε-term with the application term $z^{\delta^+}(v_0)\cdots(v_{l-1})$ of the same type as z (for a new free δ^+-variable z^{δ^+}) and to extend the choice-condition C by

$$C(z^{\delta^+}) \quad := \quad \lambda v_0.\ \ldots \lambda v_{l-1}.\ \big(\ A\{z \mapsto z^{\delta^+}(v_0)\cdots(v_{l-1})\}\ \big).$$

Example 4.5 (Higher-Order Choice-Condition) *(continuing Example 4.3 of § 4.3)*

In our framework, the complete elimination of ε-terms in (4.3.1) of Example 4.3 results in

$$\mathsf{P}(x_a^{\delta^+}, y_c^{\delta^+}, z_d^{\delta^+}) \qquad \text{(cf. (4.3.1)!)}$$

with the following higher-order choice-condition:

$$\begin{array}{rcll}
C(z_d^{\delta^+}) & := & \neg\mathsf{P}(x_a^{\delta^+}, y_c^{\delta^+}, z_d^{\delta^+}) & \text{(cf. (4.3.2)!)} \\
C(y_c^{\delta^+}) & := & \mathsf{P}(x_a^{\delta^+}, y_c^{\delta^+}, z_c^{\delta^+}(y_c^{\delta^+})) & \text{(cf. (4.3.3)!)} \\
C(z_c^{\delta^+}) & := & \lambda y.\ \neg\mathsf{P}(x_a^{\delta^+}, y, z_c^{\delta^+}(y)) & \text{(cf. (4.3.4)!)} \\
C(x_a^{\delta^+}) & := & \neg\mathsf{P}(x_a^{\delta^+}, y_a^{\delta^+}(x_a^{\delta^+}), z_b^{\delta^+}(x_a^{\delta^+})) & \text{(cf. (4.3.5)!)} \\
C(z_b^{\delta^+}) & := & \lambda x.\ \neg\mathsf{P}(x, y_a^{\delta^+}(x), z_b^{\delta^+}(x)) & \text{(cf. (4.3.6)!)} \\
C(y_a^{\delta^+}) & := & \lambda x.\ \mathsf{P}(x, y_a^{\delta^+}(x), z_a^{\delta^+}(x)(y_a^{\delta^+}(x))) & \text{(cf. (4.3.7)!)} \\
C(z_a^{\delta^+}) & := & \lambda x.\ \lambda y.\ \neg\mathsf{P}(x, y, z_a^{\delta^+}(x)(y)) & \text{(cf. (4.3.8)!)}
\end{array}$$

Notice that this representation of (4.3.1) is smaller and easier to understand than all previous ones. Indeed, by combination of λ-abstraction and term sharing via free δ^+-variables, in our framework the ε becomes practically feasible for the first time.

4.6 Instantiating Free δ^+-Variables ("ε-Substitution")

Having realized Requirement I (Syntax) of § 1 in the previous § 4.5, in this § 4.6 we are now going to explain how to satisfy Requirement II (Reasoning). To this end, we have to explain how to replace free δ^+-variables with terms that satisfy their choice-conditions.

The first thing to know about free δ^+-variables is: Just like the free γ-variables and contrary to free δ^--variables, the free δ^+-variables are *rigid* in the sense that the only way to replace a free δ^+-variable is to do it *globally*, i.e. in all formulas and all choice-conditions in an atomic transaction.

In *reductive* theorem proving such as in sequent, tableau, or matrix calculi we are in the following situation: While a free γ-variable x^{γ} can be replaced with nearly everything, the replacement of a free δ^+-variable y^{δ^+} requires some proof work, and a free δ^--variable z^{δ^-} cannot be instantiated at all.

Contrariwise, when formulas are used as tools instead of tasks, free δ^--variables can indeed be replaced—and this even locally (i.e. non-rigidly). This is the case not only for purely *generative* calculi, such as resolution and paramodulation calculi and Hilbert-style calculi such as the predicate calculus of [Hilbert & Bernays, 1968/70], but also for the lemma and induction hypothesis application in the otherwise reductive calculi of [Wirth, 2004], cf. [Wirth, 2004, § 2.5.2].

More precisely—again considering *reductive* theorem proving, where formulas are proof tasks—a free γ-variable x^{γ} may be instantiated with any term (of appropriate type) that does not violate the current variable-condition, cf. § 5.2 for details. The instantiation of a free δ^+-variable y^{δ^+} additionally requires some proof work depending on the current choice-condition C, which also puts some requirements on the variable-condition R and thus is formally called an *R-choice-condition*, cf. Definition 5.9 for the formal details. In general, if a substitution σ replaces—possibly among other free γ-variables and free δ^+-variables—the free δ^+-variable y^{δ^+} in the domain of the R-choice-condition C, then—to know that the global instantiation of the whole proof forest with σ preserves its soundness—we have to prove $(Q_C(y^{\delta^+}))\sigma$, where Q_C is given as follows:

Definition 4.6 (Q_C)
For an R-choice-condition C, we let Q_C be a total function from $\mathrm{dom}(C)$ into the set of single-formula sequents such that for each $y^{\delta^+} \in \mathrm{dom}(C)$ with $C(y^{\delta^+}) = \lambda v_0. \ldots \lambda v_{l-1}. B$ for a formula B, we have

$$Q_C(y^{\delta^+}) \qquad = \qquad \forall v_0. \ldots \forall v_{l-1}. \Big(\ \exists y. \ (B\{y^{\delta^+}(v_0)\cdots(v_{l-1}) \mapsto y\}) \quad \Rightarrow \quad B \ \Big)$$

for an arbitrary fresh bound variable $y \in \mathrm{V}_{\mathrm{bound}} \backslash \mathcal{V}(C(y^{\delta^+}))$.

Note that $Q_C(y^{\delta^+})$ is nothing but a formulation of axiom (ε_0) from § 2.1.3 in our framework, and Lemma 5.19 states its validity.

It is an essential[8] property of our choice-conditions that all occurrences of y^{δ^+} in B necessarily are of the form $y^{\delta^+}(v_0)\cdots(v_{l-1})$, cf. Definition 5.9(2). Therefore, the formula $Q_C(y^{\delta^+})$ is logically equivalent to the formula

$$\forall v_0. \ldots \forall v_{l-1}. \Big(\ \exists z. \ (B\{y^{\delta^+} \mapsto z\}) \quad \Rightarrow \quad B \ \Big)$$

for a new bound variable z of the same type as y^{δ^+}.

Example 4.7 (Predecessor Function)
Suppose that our domain is natural numbers and that $y^{\delta^+}_{(\mathsf{p1})}$ has the choice-condition

$$C(y^{\delta^+}_{(\mathsf{p1})}) = \lambda v.\ (v = y^{\delta^+}_{(\mathsf{p1})}(v){+}1).$$

Then, before we may instantiate $y^{\delta^+}_{(\mathsf{p1})}$ with the symbol p for the predecessor function specified by $\forall x.\ (\mathsf{p}(x{+}1) = x)$, we have to prove $(Q(y^{\delta^+}_{(\mathsf{p1})}))\{y^{\delta^+}_{(\mathsf{p1})} \mapsto \mathsf{p}\}$, which reads as

$$\forall v.\ \big(\ \exists y.\ (v = y{+}1) \Rightarrow (v = \mathsf{p}(v){+}1)\ \big),$$

and is valid in arithmetic.

Example 4.8 (Canossa 1077) *(continuing Example 2.4)*

The situation of Example 2.4 now reads

$$\mathsf{Holy\ Ghost} \;=\; z^{\delta^+}_0 \qquad \wedge \qquad \mathsf{Joseph} \;=\; z^{\delta^+}_1 \tag{4.8.1}$$

with $\qquad C(z^{\delta^+}_0) = \mathsf{Father}(z^{\delta^+}_0, \mathsf{Jesus}),$
and $\qquad C(z^{\delta^+}_1) = \mathsf{Father}(z^{\delta^+}_1, \mathsf{Jesus}).$

This does not bring us into the old trouble with the Pope because nobody knows whether $z^{\delta^+}_0 = z^{\delta^+}_1$ holds.

On the one hand, knowing (2.1.2) from Example 2.1 of § 2.1, we can prove (4.8.1) as follows: We first substitute $z^{\delta^+}_0$ with $\mathsf{Holy\ Ghost}$ because, for $\sigma_0 := \{z^{\delta^+}_0 {\mapsto} \mathsf{Holy\ Ghost}\}$, we have $(C(z^{\delta^+}_0))\sigma_0$ and—a fortiori— $(Q_C(z^{\delta^+}_0))\sigma_0$, which reads

$$\exists z.\ \mathsf{Father}(z, \mathsf{Jesus}) \quad \Rightarrow \quad \mathsf{Father}(\mathsf{Holy\ Ghost}, \mathsf{Jesus});$$

and, analogously, substitute $z^{\delta^+}_1$ with Joseph because, for $\sigma_1 := \{z^{\delta^+}_1 {\mapsto} \mathsf{Joseph}\}$, we have $(C(z^{\delta^+}_1))\sigma_1$ and—a fortiori— $(Q_C(z^{\delta^+}_1))\sigma_1$. After these substitutions, (4.8.1) becomes the tautology

$$\mathsf{Holy\ Ghost} \;=\; \mathsf{Holy\ Ghost} \qquad \wedge \qquad \mathsf{Joseph} \;=\; \mathsf{Joseph}$$

On the other hand, if we want to have trouble, we can apply the substitution

$$\sigma' \;=\; \{z^{\delta^+}_0 \mapsto \mathsf{Joseph},\ z^{\delta^+}_1 \mapsto \mathsf{Joseph}\}$$

to (4.8.1) because of $(Q_C(z^{\delta^+}_0))\sigma' = (Q_C(z^{\delta^+}_1))\sigma_1 = (Q_C(z^{\delta^+}_1))\sigma'$.
Then our task is to show

$$\mathsf{Holy\ Ghost} \;=\; \mathsf{Joseph} \qquad \wedge \qquad \mathsf{Joseph} \;=\; \mathsf{Joseph}$$

Note that this procedure is stupid already under the aspect of theorem proving alone.

5 Formal Presentation of Our Indefinite Semantics

To satisfy Requirement III (Semantics) of § 1, in this § 5 we present our novel semantics for the ε *formally*. This is required for precision and consistency. As consistency of our new semantics is not trivial at all, technical rigor cannot be avoided. From § 4 the reader should have a good intuition of our intended representation and semantics of the ε, free δ^+-variables, and choice-conditions in our framework. § 5 organizes as follows: In § 5.2 and § 5.4 we formalize variable-conditions and explain how to deal with free γ-variables syntactically and semantically. In § 5.3 we introduce a preliminary semantics that does not treat free δ^+-variables properly, and in § 5.6 the proper semantics. Only between these two §§ 5.3 and 5.6, we can discuss choice-conditions (§ 5.5). Our interest goes beyond soundness in that we want "*preservation of solutions*". By this we mean the following: All *closing substitutions* for the free γ-variables and free δ^+-variables—i.e. all solutions that transform a proof attempt (to which a proposition has been reduced) into a closed proof—are also solutions of the original proposition. This is similar to a proof in Prolog, computing answers to a query proposition that contains free γ-variables. Therefore, in § 5.7 we discuss this solution-preserving notion of *reduction*, especially under the aspect of global instantiation of free δ^+-variables. Finally, in § 5.8 we give some hints on the design of operators similar to our ε. All in all, in this § 5, we extend and simplify the presentation of [Wirth, 2004], which, however, additionally contains comparative discussions, compatible extensions for *descente infinie*, and those proofs that are omitted here.

5.1 Basic Notions and Notation

'$\mathbf{N}$' denotes the set of natural numbers and '$\prec$' the ordering on $\mathbf{N}$. Let $\mathbf{N}_+ := \{\, n \in \mathbf{N} \mid 0 \neq n \,\}$. We use '$\uplus$' for the union of disjoint classes and 'id' for the identity function. For classes R, A, and B we define:

$\mathrm{dom}(R)$	$:= \{\, a \mid \exists b.\ (a,b) \in R \,\}$	*domain*
${}_A{\upharpoonleft}R$	$:= \{\, (a,b) \in R \mid a \in A \,\}$	*restriction to* A
$\langle A \rangle R$	$:= \{\, b \mid \exists a \in A.\ (a,b) \in R \,\}$	*image of* A, i.e. $\langle A \rangle R = \mathrm{ran}({}_A{\upharpoonleft}R)$

And the dual ones:

$\mathrm{ran}(R)$	$:= \{\, b \mid \exists a.\ (a,b) \in R \,\}$	*range*
$R{\upharpoonright}_B$	$:= \{\, (a,b) \in R \mid b \in B \,\}$	*range-restriction to* B
$R\langle B \rangle$	$:= \{\, a \mid \exists b \in B.\ (a,b) \in R \,\}$	*reverse-image of* B, i.e. $R\langle B \rangle = \mathrm{dom}(R{\upharpoonright}_B)$

Furthermore, we use '$\emptyset$' to denote the empty set as well as the empty function. Functions are (right-) unique relations and the meaning of '$f{\circ}g$' is extensionally given by $(f{\circ}g)(x) = g(f(x))$. The *class of total functions from* A *to* B is denoted as $A \rightarrow B$. The *class of (possibly) partial functions from* A *to* B is denoted as $A \rightsquigarrow B$. Both $\rightarrow$ and $\rightsquigarrow$ associate to the right, i.e. $A \rightsquigarrow B \rightarrow C$ reads $A \rightsquigarrow (B \rightarrow C)$.

Let R be a binary relation. R is said to be a relation *on* A if $\mathrm{dom}(R) \cup \mathrm{ran}(R) \subseteq A$. R is *irreflexive* if $\mathrm{id}{\cap}R = \emptyset$. It is A-*reflexive* if ${}_A{\upharpoonleft}\mathrm{id} \subseteq R$. Speaking of a *reflexive* relation we refer to the largest A that is appropriate in the local context, and referring to this A we write R^0 to ambiguously denote ${}_A{\upharpoonleft}\mathrm{id}$. With $R^1 := R$, and $R^{n+1} := R^n{\circ}R$ for $n \in \mathbf{N}_+$, R^m denotes the m-step relation for R. The *transitive closure* of R is $R^+ := \bigcup_{n \in \mathbf{N}_+} R^n$. The *reflexive & transitive closure* of R is $R^* := \bigcup_{n \in \mathbf{N}} R^n$. A relation R (on A) is *well-founded* if any non-empty class B ($\subseteq A$) has an R-minimal element, i.e. $\exists a \in B.\ \neg\exists a' \in B.\ a' R\, a$.

5.2 Variables and R-Substitutions

We assume the following four sets of symbols to be disjoint:

V_γ	*free γ-variables*, i.e. the free variables of [Fitting, 1996]
V_δ	*free δ-variables*, i.e. nullary parameters, instead of Skolem functions
$\mathrm{V}_{\mathrm{bound}}$	*bound variables*, i.e. variables to be bound, cf. below
Σ	*constants*, i.e. the function and predicate symbols from the signature

As explained in § 4.1, we partition the free δ-variables into *free δ^--variables* and *free δ^+-variables*: $\mathrm{V}_\delta = \mathrm{V}_{\delta^-} \uplus \mathrm{V}_{\delta^+}$. We define the *free variables* by $\mathrm{V}_{\mathrm{free}} := \mathrm{V}_\gamma \uplus \mathrm{V}_\delta$ and the *variables* by $\mathrm{V} := \mathrm{V}_{\mathrm{bound}} \uplus \mathrm{V}_{\mathrm{free}}$. Finally, the *rigid* variables by $\mathrm{V}_{\gamma\delta^+} := \mathrm{V}_\gamma \uplus \mathrm{V}_{\delta^+}$. We use '$\mathcal{V}_k(\Gamma)$' to denote the set of variables from V_k occurring in Γ.

Let σ be a substitution. σ is a *substitution on X* if $\mathrm{dom}(\sigma) \subseteq X$. We denote with '$\Gamma\sigma$' the result of replacing each occurrence of a variable $x \in \mathrm{dom}(\sigma)$ in Γ with $\sigma(x)$. (Actually, we may have to rename some of the bound variables in $\sigma(x)$ when we exclude the binding of a variable within the scope of a bound variable of the same name.) Unless otherwise stated, we tacitly assume that all occurrences of variables from $\mathrm{V}_{\mathrm{bound}}$ in a term or formula or in the range of a substitution are *bound occurrences* (i.e. that a variable $x \in \mathrm{V}_{\mathrm{bound}}$ occurs only in the scope of a binder on x) and that each substitution σ satisfies $\mathrm{dom}(\sigma) \subseteq \mathrm{V}_{\mathrm{free}}$, so that no bound occurrences of variables can be replaced and no additional variable occurrences can become bound (i.e. captured) when applying σ.

Several binary relations on free variables will be introduced in this and the following §§. The overall idea is that when (x, y) occurs in such a relation this means something like "x is necessarily older than y" or "the value of y depends on x or is described in terms of x".

Definition 5.1 (Variable-Condition) A *variable-condition* is a subset of $\mathrm{V}_{\mathrm{free}} \times \mathrm{V}_{\mathrm{free}}$.

Definition 5.2 (σ-Update) Let R be a variable-condition and σ be a substitution. The *σ-update of R* is $R \;\cup\; \{\, (z^{\mathrm{free}}, x^{\mathrm{free}}) \mid x^{\mathrm{free}} \in \mathrm{dom}(\sigma) \wedge z^{\mathrm{free}} \in \mathcal{V}_{\mathrm{free}}(\sigma(x^{\mathrm{free}})) \,\}$.

Definition 5.3 (R-Substitution) Let R be a variable-condition.
σ is an *R-substitution* if σ is a substitution and the σ-update of R is well-founded.

Syntactically, $(x^{\mathrm{free}}, y^{\mathrm{free}}) \in R$ is to express that an R-substitution σ must not replace x^{free} with a term in which y^{free} could ever occur. This is guaranteed when the σ-updates R' of R are always required to be well-founded. For $z^{\mathrm{free}} \in \mathcal{V}_{\mathrm{free}}(\sigma(x^{\mathrm{free}}))$, we get $z^{\mathrm{free}}\; R'\; x^{\mathrm{free}}\; R'\; y^{\mathrm{free}}$, blocking z^{free} against terms containing y^{free}. Note that in practice a σ-update of R can always be chosen to be finite. In this case, it is well-founded iff it is acyclic.

5.3 R-Validity

Instead of defining validity from scratch, we require some abstract properties typically holding in two-valued semantics. Validity is given relative to some Σ-structure $\mathcal{S}$, assigning a non-empty universe (or "carrier") to each type. For $\mathrm{X} \subseteq \mathrm{V}$ we denote the set of total

$\mathcal{S}$-valuations of X (i.e. functions mapping variables to objects of the universe of $\mathcal{S}$ (respecting types)) with

$$\mathrm{X} \rightarrow \mathcal{S}$$

and the set of (possibly) partial $\mathcal{S}$-valuations of X with

$$\mathrm{X} \rightsquigarrow \mathcal{S}$$

For $\delta : \mathrm{X} \rightarrow \mathcal{S}$ we denote with '$\mathcal{S}\uplus\delta$' the extension of $\mathcal{S}$ to the variables of X. More precisely, we assume some evaluation function 'eval' such that $\mathrm{eval}(\mathcal{S}\uplus\delta)$ maps any term whose constants and freely occurring variables are from $\Sigma\uplus\mathrm{X}$ into the universe of $\mathcal{S}$ (respecting types) such that for all $x \in \mathrm{X}$: $\mathrm{eval}(\mathcal{S}\uplus\delta)(x) = \delta(x)$. Moreover, $\mathrm{eval}(\mathcal{S}\uplus\delta)$ maps any formula B whose constants and freely occurring variables are from $\Sigma\uplus\mathrm{X}$ to TRUE or FALSE, such that B is valid in $\mathcal{S}\uplus\delta$ iff $\mathrm{eval}(\mathcal{S}\uplus\delta)(B) = \mathsf{TRUE}$.

Notice that we leave open what our formulas and what our Σ-structures exactly are. The latter can range from a first-order Σ-structure to a higher-order modal Σ-model, provided that the following two standard textbook lemmas hold for a term or formula B (possibly with some *unbound* occurrences of variables from $\mathrm{V}_{\mathrm{bound}}$) and a Σ-structure $\mathcal{S}$ with valuation $\delta : \mathrm{V} \rightsquigarrow \mathcal{S}$.

Explicitness Lemma

The value of the evaluation function on B depends only on the valuation of those variables that actually occur freely in B; formally: For X being the set of variables that occur freely in B, if $\mathrm{X} \subseteq \mathrm{dom}(\delta)$: $\mathrm{eval}(\mathcal{S}\uplus\delta)(B) = \mathrm{eval}(\mathcal{S} \uplus {}_{\mathrm{X}}{\restriction}\delta)(B)$.

Substitution [Value] Lemma

Let σ be a substitution. If the variables that occur freely in $B\sigma$ belong to $\mathrm{dom}(\delta)$, then:

$$\mathrm{eval}(\mathcal{S}\uplus\delta)(B\sigma) = \mathrm{eval}\Big(\mathcal{S} \uplus \big((\sigma \uplus {}_{\mathrm{V}\setminus\mathrm{dom}(\sigma)}{\restriction}\mathrm{id}) \circ \mathrm{eval}(\mathcal{S}\uplus\delta) \big) \Big)\Big(B \Big).$$

We are now going to define a new notion of validity of sets of sequents, i.e. sets of lists of formulas. As this new kind of validity depends on a variable-condition R, it is called "R-validity". It provides the free γ-variables with an existential semantics given by their valuation $\epsilon(e)(\delta) : \mathrm{V}_\gamma \rightarrow \mathcal{S}$, and the free δ-variables with a universal semantics by $\delta : \mathrm{V}_\delta \rightarrow \mathcal{S}$. The definition is top-down and the function ϵ (having nothing to do with Hilbert's ε) and the notion of an $(\mathcal{S}, R)$-valuation are to be explained in § 5.4, which also contains examples illustrating R-Validity.

Definition 5.4 (R-Validity, K**)** Let R be a variable-condition. Let $\mathcal{S}$ be a Σ-structure with valuation $\delta : \mathrm{V} \rightsquigarrow \mathcal{S}$. Let G be a set of sequents.
G is *R-valid in $\mathcal{S}$* if there is an $(\mathcal{S}, R)$-valuation e such that G is $(e, \mathcal{S})$-valid.
G is *$(e, \mathcal{S})$-valid* if G is $(\delta', e, \mathcal{S})$-valid for all $\delta' : \mathrm{V}_\delta \rightarrow \mathcal{S}$.
G is *$(\delta, e, \mathcal{S})$-valid* if G is valid in $\mathcal{S} \uplus \epsilon(e)(\delta) \uplus \delta$.
G is *valid in $\mathcal{S}\uplus\delta$* if Γ is valid in $\mathcal{S}\uplus\delta$ for all $\Gamma \in G$.
A sequent Γ is *valid in $\mathcal{S}\uplus\delta$* if there is some formula listed in Γ that is valid in $\mathcal{S}\uplus\delta$.
Validity in a class of Σ-structures is understood as validity in each of the Σ-structures of that class. If we omit the reference to a special Σ-structure we mean validity in some fixed class K of Σ-structures, such as the class of all Σ-structures or the class of Herbrand Σ-structures.

5.4 $(\mathcal{S}, R)$-Valuations

Let $\mathcal{S}$ be some Σ-structure. We now define semantical counterparts of our R-substitutions on V_γ, which we will call "$(\mathcal{S}, R)$-valuations". As an $(\mathcal{S}, R)$-valuation plays the rôle of a *raising function* (a dual of a Skolem function as defined in [Miller, 1992]), it does not simply map each free γ-variable directly to an object of $\mathcal{S}$ (of the same type), but may additionally read the values of some free δ-variables under an $\mathcal{S}$-valuation $\delta : \mathrm{V}_\delta \to \mathcal{S}$. More precisely, an $(\mathcal{S}, R)$-valuation e takes some restriction of δ as a second argument, say $\delta' : \mathrm{V}_\delta \rightsquigarrow \mathcal{S}$ with $\delta' \subseteq \delta$. In short:

$$e : \mathrm{V}_\gamma \to (\mathrm{V}_\delta \rightsquigarrow \mathcal{S}) \rightsquigarrow \mathcal{S}.$$

Moreover, for each free γ-variable x^γ, we require that the set $\mathrm{dom}(\delta')$ of free δ-variables read by $e(x^\gamma)$ is identical for all δ. This identical set will be denoted with $S_e\langle\!\{x^\gamma\}\!\rangle$ below. Technically, we require that there is some "semantical relation" $S_e \subseteq \mathrm{V}_\delta \times \mathrm{V}_\gamma$ such that for all $x^\gamma \in \mathrm{V}_\gamma$:

$$e(x^\gamma) \;:\; (S_e\langle\!\{x^\gamma\}\!\rangle \to \mathcal{S}) \to \mathcal{S}.$$

This means that $e(x^\gamma)$ can read the value of y^δ if and only if $(y^\delta, x^\gamma) \in S_e$. Note that, for each $e : \mathrm{V}_\gamma \to (\mathrm{V}_\delta \rightsquigarrow \mathcal{S}) \rightsquigarrow \mathcal{S}$, at most one semantical relation exists, namely

$$S_e \;:=\; \{\ (y^\delta, x^\gamma) \mid x^\gamma \in \mathrm{V}_\gamma \wedge y^\delta \in \mathrm{dom}(\textstyle\bigcup(\mathrm{dom}(e(x^\gamma))))\ \}.$$

In some of the following definitions we are slightly more general because we want to apply the terminology not only to free γ-variables but also to free δ^+-variables.

Definition 5.5 (Semantical Relation (S_e)) The *semantical relation for* e is

$$S_e \;:=\; \{\ (y, x) \mid x \in \mathrm{dom}(e) \wedge y \in \mathrm{dom}(\textstyle\bigcup(\mathrm{dom}(e(x))))\ \}.$$

e is *semantical* if e is a partial function on V such that for all $x \in \mathrm{dom}(e)$:

$$e(x) : (S_e\langle\!\{x\}\!\rangle \to \mathcal{S}) \to \mathcal{S}.$$

Definition 5.6 ($(\mathcal{S}, R)$-Valuation)
Let R be a variable-condition and let $\mathcal{S}$ be a Σ-structure. e is an $(\mathcal{S}, R)$*-valuation* if $e : \mathrm{V}_\gamma \to (\mathrm{V}_\delta \rightsquigarrow \mathcal{S}) \rightsquigarrow \mathcal{S}$, e is semantical, and $R \cup S_e$ is well-founded.

Finally, we need the technical means to turn an $(\mathcal{S}, R)$-valuation e together with a valuation δ of the free δ-variables into a valuation $\epsilon(e)(\delta)$ of the free γ-variables:

Definition 5.7 (ϵ)
We define the function $\epsilon : \ (\mathrm{V} \rightsquigarrow (\mathrm{V} \rightsquigarrow \mathcal{S}) \rightsquigarrow \mathcal{S}) \ \to\ (\mathrm{V} \rightsquigarrow \mathcal{S}) \ \to\ \mathrm{V} \ \rightsquigarrow\ \mathcal{S}$
for $e : \mathrm{V} \rightsquigarrow (\mathrm{V} \rightsquigarrow \mathcal{S}) \rightsquigarrow \mathcal{S}$, $\quad \delta : \mathrm{V} \rightsquigarrow \mathcal{S}$, $\quad x \in \mathrm{V}$
by $\epsilon(e)(\delta)(x) := e(x)({}_{S_e\langle\!\{x\}\!\rangle}{\upharpoonright}\delta)$.

Example 5.8 (R-Validity) For $x^\gamma \in \mathrm{V}_\gamma$, $y^\delta \in \mathrm{V}_\delta$, the sequent $x^\gamma{=}y^\delta$ is $\emptyset$-valid in any $\mathcal{S}$ because we can choose $S_e := \mathrm{V}_\delta{\times}\mathrm{V}_\gamma$ and $e(x^\gamma)(\delta) := \delta(y^\delta)$ for $\delta : \mathrm{V}_\delta \to \mathcal{S}$, resulting in $\epsilon(e)(\delta)(x^\gamma) = e(x^\gamma)({}_{S_e\langle\{x^\gamma\}\rangle}{\upharpoonleft}\delta) = e(x^\gamma)({}_{\mathrm{V}_\delta}{\upharpoonleft}\delta) = \delta(y^\delta)$. This means that $\emptyset$-validity of $x^\gamma{=}y^\delta$ is the same as validity of $\forall y.\ \exists x.\ x{=}y$. Moreover, note that $\epsilon(e)(\delta)$ has access to the δ-value of y^δ just as a raising function f for x in the raised (i.e. dually Skolemized) version $f(y^\delta){=}y^\delta$ of $\forall y.\ \exists x.\ x{=}y$.

Contrary to this, for $R := \mathrm{V}_\gamma{\times}\mathrm{V}_\delta$, the same formula $x^\gamma{=}y^\delta$ is not R-valid in general because then the required well-foundedness of $R \cup S_e$ (cf. Definition 5.6) implies $S_e = \emptyset$, and the value of x^γ cannot depend on $\delta(y^\delta)$ anymore, due to $e(x^\gamma)({}_{S_e\langle\{x^\gamma\}\rangle}{\upharpoonleft}\delta) = e(x^\gamma)({}_{\emptyset}{\upharpoonleft}\delta) = e(x^\gamma)(\emptyset)$. This means that $(\mathrm{V}_\gamma{\times}\mathrm{V}_\delta)$-validity of $x^\gamma{=}y^\delta$ is the same as validity of $\exists x.\ \forall y.\ x{=}y$. Moreover, note that $\epsilon(e)(\delta)$ has no access to the δ-value of y^δ just as a raising function c for x in the raised version $c{=}y^\delta$ of $\exists x.\forall y.\,x{=}y$.

For a more general example let $G = \{\ A_{i,0} \ldots A_{i,n_i-1} \mid i \in I\ \}$, where for $i \in I$ and $j \prec n_i$ the $A_{i,j}$ are formulas with free γ-variables from $\boldsymbol{e}$ and free δ-variables from $\boldsymbol{u}$. Then
$(\mathrm{V}_\gamma{\times}\mathrm{V}_\delta)$-validity of G means $\quad \exists \boldsymbol{e}.\ \forall \boldsymbol{u}.\ \forall i \in I.\ \exists j \prec n_i.\ A_{i,j}$
whereas $\emptyset$-validity of G means $\quad \forall \boldsymbol{u}.\ \exists \boldsymbol{e}.\ \forall i \in I.\ \exists j \prec n_i.\ A_{i,j}$

Also any other sequence of universal and existential quantifiers can be represented by a variable-condition R, starting from the empty set and applying the δ-rules from § 4.2. A translation of a variable-condition R into a sequence of quantifiers may, however, require a strengthening of dependences, in the sense that a backwards translation would result in a variable-condition R' with $R \subsetneq R'$. This means that our framework can express logical dependences more fine-grained than standard quantifiers.

5.5 Choice-Conditions

Definition 5.9 (Choice-Condition)

C is an *R-choice-condition* if R is a well-founded variable-condition and C is a partial function from V_{δ^+} into the set of formula-valued λ-terms, such that for all $y^{\delta^+} \in \mathrm{dom}(C)$:

1. $z^{\mathrm{free}}\ R^*\ y^{\delta^+}$ for all $z^{\mathrm{free}} \in \mathcal{V}_{\mathrm{free}}(C(y^{\delta^+}))$, and
2. $C(y^{\delta^+})$ is of the form $\lambda v_0.\ \ldots \lambda v_{l-1}.\ B$, where

   B is a formula whose freely occurring variables from $\mathrm{V}_{\mathrm{bound}}$ are among $\{v_0, \ldots, v_{l-1}\} \subseteq \mathrm{V}_{\mathrm{bound}}$

   and where, for $v_0 : \alpha_0, \quad \ldots, \quad v_{l-1} : \alpha_{l-1},$ we have

   $$y^{\delta^+} : \alpha_0 \to \cdots \to \alpha_{l-1} \to \alpha_l \quad \text{for some type } \alpha_l,$$

   and any occurrence of y^{δ^+} in B is of the form $y^{\delta^+}(v_0)\cdots(v_{l-1})$.

Example 5.10 (Choice-Condition) *(continuing Example 4.5)*

(a) If R is a well-founded variable-condition that satisfies

$$z_a^{\delta^+} \; R \; y_a^{\delta^+} \; R \; z_b^{\delta^+} \; R \; x_a^{\delta^+} \; R \; z_c^{\delta^+} \; R \; y_c^{\delta^+} \; R \; z_d^{\delta^+},$$

then the C of Example 4.5 is an R-choice-condition, indeed.

(b) If some clever person would like to do the complete quantifier elimination of Example 4.5 by

$$\begin{array}{lcl} C'(z_d^{\delta^+}) & := & \neg\mathsf{P}(x_a^{\delta^+}, y_c^{\delta^+}, z_d^{\delta^+}) \\ C'(y_c^{\delta^+}) & := & \;\;\mathsf{P}(x_a^{\delta^+}, y_c^{\delta^+}, z_d^{\delta^+}) \\ C'(x_a^{\delta^+}) & := & \neg\mathsf{P}(x_a^{\delta^+}, y_c^{\delta^+}, z_d^{\delta^+}) \end{array}$$

then he would—among other things—need $z_d^{\delta^+} \; R^+ \; y_c^{\delta^+} \; R^+ \; z_d^{\delta^+}$, by Definition 5.9(1) due to the values of C' at $y_c^{\delta^+}$ and $z_d^{\delta^+}$. This renders R non-well-founded. Thus, this C' cannot be an R-choice-condition for any R. Note that the choices required by C' for $y_c^{\delta^+}$ and $z_d^{\delta^+}$ are in an unsolvable conflict, indeed.

(c) For a more elementary example, take

$$C''(x^{\delta^+}) \;\; := \;\; (x^{\delta^+} = y^{\delta^+}) \qquad\qquad C''(y^{\delta^+}) \;\; := \;\; (x^{\delta^+} \neq y^{\delta^+})$$

Then x^{δ^+} and y^{δ^+} form a vicious circle of conflicting choices for which no valuation can be found that is compatible with C'', cf. Definition 5.11, Lemma 5.12. But C'' is no choice-condition at all because there is no well-founded variable-condition R that could turn it into an R-choice-condition.

We now split our valuation $\delta : \mathrm{V}_\delta \to \mathcal{S}$; while $\tau : \mathrm{V}_{\delta^-} \to \mathcal{S}$ valuates the free δ^--variables, π valuates the remaining free δ^+-variables. As the choices of π may depend on τ, the technical realization is similar to that of the dependence of the $(\mathcal{S}, R)$-valuations on the free δ-variables, as described in § 5.4.

Definition 5.11 (Compatibility)

Let C be an R-choice-condition, $\mathcal{S}$ a Σ-structure, and e an $(\mathcal{S}, R)$-valuation. π is $(e, \mathcal{S})$-*compatible with* (C, R) if

1. $\pi : \mathrm{V}_{\delta^+} \to (\mathrm{V}_{\delta^-} \rightsquigarrow \mathcal{S}) \rightsquigarrow \mathcal{S}$ is semantical (cf. Definition 5.5) and $R \cup S_e \cup S_\pi$ is well-founded.

2. For all $y^{\delta^+} \in \mathrm{dom}(C)$ with $C(y^{\delta^+}) = \lambda v_0. \; \ldots \lambda v_{l-1}. \; B$ for a formula B, for all $\tau : \mathrm{V}_{\delta^-} \to \mathcal{S}$, for all $\eta : \{y^{\delta^+}\} \to \mathcal{S}$, and for all $\chi : \{v_0, \ldots, v_{l-1}\} \to \mathcal{S}$, setting $\delta := \epsilon(\pi)(\tau) \uplus \tau \uplus \chi$ and $\delta' := \eta \uplus {}_{\mathrm{V} \setminus \{y^{\delta^+}\}}\!\upharpoonright\!\delta$ (i.e. δ' is the η-variant of δ):

   If B is $(\delta', e, \mathcal{S})$-valid, then B is also $(\delta, e, \mathcal{S})$-valid.

Roughly speaking, Item 1 of this definition requires—for similar reasons as before—that the flow of information between variables expressed in R, e, and π is acyclic.

To understand Item 2, consider an R-choice-condition $C := \{(y^{\delta^+},\ \lambda v_0.\ \ldots\lambda v_{l-1}.\ B)\}$, which restricts the value of y^{δ^+} with the formula-valued λ-term $\lambda v_0.\ \ldots\lambda v_{l-1}.\ B$. Then C simply requires that a different choice for the $\epsilon(\pi)(\tau)$-value of y^{δ^+} cannot give rise to the validity of the formula B in $\mathcal{S} \uplus \epsilon(e)(\delta) \uplus \delta$. Or—in other words—that $\epsilon(\pi)(\tau)(y^{\delta^+})$ is chosen such that B becomes valid, whenever such a choice is possible. This is closely related to Hilbert's ε-operator in the sense that y^{δ^+} is given the value of

$$\lambda v_0.\ \ldots\lambda v_{l-1}.\ \varepsilon y.\ (B\{y^{\delta^+}(v_0)\cdots(v_{l-1}) \mapsto y\})$$

for a fresh bound variable y.

As the choice for y^{δ^+} depends on the other free variables of $\lambda v_0.\ \ldots\lambda v_{l-1}.\, B$ (i.e. the free variables of $\lambda v_0.\ \ldots\lambda v_{l-1}.\ \varepsilon y.\ (B\{y^{\delta^+}(v_0)\cdots(v_{l-1}) \mapsto y\})$), we included this dependence into the transitive closure of the variable-condition R in Definition 5.9(1). Therefore, the well-foundedness of R avoids the conflict of Example 5.10(c).

Note that the empty function $\emptyset$ is an R-choice-condition for any well-founded variable-condition R. Furthermore, any π with $\pi : \mathrm{V}_{\delta^+} \to \{\emptyset\} \to \mathcal{S}$ is $(e,\mathcal{S})$-compatible with $(\emptyset, R)$ due to $S_\pi = \emptyset$. Indeed, as stated in the following lemma, a compatible π always exists. This is due to Definition 5.9(1) and the well-foundedness of $R \cup S_e$ (according to Definition 5.6) and due to the restriction on the occurrence of y^{δ^+} in B in Definition 5.9(2).

Lemma 5.12 *If C is an R-choice-condition, $\mathcal{S}$ a Σ-structure, and e an $(\mathcal{S},R)$-valuation, then there is some π that is $(e,\mathcal{S})$-compatible with (C,R).*

Just like the variable-condition R, the R-choice-condition C may grow during proofs. This kind of extension together with a simple soundness condition plays an important rôle in inference:

Definition 5.13 (Extension) (C',R') is an *extension of* (C,R) if C is an R-choice-condition, C' is an R'-choice-condition, $C \subseteq C'$, and $R \subseteq R'$.

Lemma 5.14 (Extension) *Let (C',R') be an extension of (C,R).*
If e is an $(\mathcal{S},R')$-valuation and π is $(e,\mathcal{S})$-compatible with (C',R'),
then e is also an $(\mathcal{S},R)$-valuation and π is also $(e,\mathcal{S})$-compatible with (C,R).

After global application of an R-substitution σ we now have to update both R and C:

Definition 5.15 (Extended σ-Update) Let C be an R-choice-condition and let σ be a substitution. The *extended σ-update* (C',R') *of* (C,R) is given by:

$$C' := \{\ (x, B\sigma) \mid (x,B) \in C \wedge x \notin \mathrm{dom}(\sigma)\ \},$$

R' is the σ-update of R, cf. Definition 5.2.

Lemma 5.16 (Extended σ-Update) *If C is an R-choice-condition, σ an R-substitution, and if (C',R') is the extended σ-update of (C,R), then C' is an R'-choice-condition.*

5.6 (C,R)-Validity

While the notion of R-validity (cf. Definition 5.4) already provides the free γ-variables with an existential semantics, it fails to give the free δ^{+}-variables the proper semantics according to an R-choice-condition C. This deficiency is overcome in the following notion of "(C,R)-validity", which—roughly speaking—requires the following: For arbitrary values of the free δ^{-}-variables, we must be able to choose values for the free δ^{+}-variables satisfying C, and then we must be able to choose values for the free γ-variables, such that the sequents become valid. Note that the dependences of these choices are restricted by R. In a formal top down representation, this reads:

Definition 5.17 ((C,R)-Validity) Let C be an R-choice-condition, let $\mathcal{S}$ be a Σ-structure, and let G be a set of sequents. G is (C,R)-*valid in* $\mathcal{S}$ if G is $(\pi,e,\mathcal{S})$-valid for some $(\mathcal{S},R)$-valuation e and some π that is $(e,\mathcal{S})$-compatible with (C,R). G is $(\pi,e,\mathcal{S})$-*valid* if G is $\big(\ \epsilon(\pi)(\tau)\uplus\tau,\quad e,\quad \mathcal{S}\ \big)$-valid for each $\tau : \mathrm{V}_{\delta^-}\to\mathcal{S}$.

Notice that the notion of $(\pi,e,\mathcal{S})$-validity with $\pi : \mathrm{V}_{\delta^+}\to(\mathrm{V}_{\delta^-}\leadsto\mathcal{S})\leadsto\mathcal{S}$ differs from $(\delta,e,\mathcal{S})$-validity with $\delta : \mathrm{V}\leadsto\mathcal{S}$ as given in Definition 5.4. Notice that (C,R)-validity treats the free δ^{+}-variables properly, whereas R-validity of Definition 5.4 does not.

In our framework the formula (E2) of § 3.1.1 looks like (E2′) in the following lemma.

Lemma 5.18 ((C,R)-Validity of (E2′)) *Let C be an R-choice-condition. For $i\in\{0,1\}$, let A_i be a formula and $x_i^{\delta^+}\in\mathrm{V}_{\delta^+}$ with $C(x_i^{\delta^+})=A_i\{x\mapsto x_i^{\delta^+}\}$, $x_i^{\delta^+}\notin\mathcal{V}(A_0,A_1)$, $x_i^{\delta^+}\notin\mathrm{dom}(R)$, $x_i^{\delta^+}\notin\mathcal{V}(\langle\mathrm{V}\setminus\{x_i^{\delta^+}\}\rangle C)$. The formula*

$$\forall x.\ (A_0\Leftrightarrow A_1)\quad\Rightarrow\quad x_0^{\delta^+}=x_1^{\delta^+}\tag{E2′}$$

is (C,R)-valid.

Note that the conditions of Lemma 5.18 may simply be achieved by taking *fresh* free δ^{+}-variables $x_0^{\delta^+}$ and $x_1^{\delta^+}$ and adding $(\mathcal{V}_{\mathrm{free}}(A_i\{x\mapsto x_i^{\delta^+}\})\setminus\{x_i^{\delta^+}\})\times\{x_i^{\delta^+}\}$ to the current variable-condition. Very roughly speaking, Lemma 5.18 holds because after choosing a value for $x_0^{\delta^+}$ we can take the same value for $x_1^{\delta^+}$, simply because $x_1^{\delta^+}$ is new and can read all free δ^{-}-variables, and especially those that $x_0^{\delta^+}$ reads. We will actually do two proofs of Lemma 5.18. First, as an exercise for the reader, a semantical one right now, which is complicated and ugly. And then in Example 5.23 a formal, nice, and short one in our calculus.

Proof of Lemma 5.18 Formally, in this proof we would have to apply the Explicitness and the Substitution [Value] Lemma from § 5.3 several times, but we just argue informally in a straightforward and intuitively clear manner. Otherwise the proof would be even longer and more ugly.

Let $B := (\forall x.\ (A_0 \Leftrightarrow A_1)\ \Rightarrow\ x_0^{\delta^+} = x_1^{\delta^+})$. Let $\mathcal{S}$ be an arbitrary Σ-structure. As universes are non-empty, there is some $(\mathcal{S}, R)$-valuation e with $S_e = \emptyset$.
By Lemma 5.12 there is some π that is $(e, \mathcal{S})$-compatible with (C, R). Define π' by

$$\pi'(x_1^{\delta^+})(\tau) := \left\{\begin{array}{ll} \epsilon(\pi)(\tau)(x_0^{\delta^+}) & \text{if } \forall x.\ (A_0 \Leftrightarrow A_1) \text{ is } (\epsilon(\pi)(\tau) \uplus \tau, e, \mathcal{S})\text{-valid.} \\ \epsilon(\pi)(\tau)(x_1^{\delta^+}) & \text{otherwise} \end{array}\right\} \text{ for } \tau : \mathrm{V}_{\delta^-} \to \mathcal{S},$$

and $\pi'(y^{\delta^+}) := \pi(y^{\delta^+})$ for all $y^{\delta^+} \in \mathrm{V}_{\delta^+} \backslash \{x_1^{\delta^+}\}$. By $x_1^{\delta^+} \notin \mathcal{V}(A_0, A_1)$, we have $\epsilon(\pi')(\tau)(x_0^{\delta^+}) = \epsilon(\pi')(\tau)(x_1^{\delta^+})$ for all $\tau : \mathrm{V}_{\delta^-} \to \mathcal{S}$ for which $\forall x.\ (A_0 \Leftrightarrow A_1)$ is $(\epsilon(\pi')(\tau) \uplus \tau, e, \mathcal{S})$-valid. Thus, B is $(\epsilon(\pi')(\tau) \uplus \tau, e, \mathcal{S})$-valid. It remains to show that π' is $(e, \mathcal{S})$-compatible with (C, R), too. As π' is obviously semantical, for Item 1 of Definition 5.11 it suffices to show that $R \cup S_e \cup S_{\pi'}$ is well-founded. By $x_1^{\delta^+} \notin \mathrm{dom}(R)$, due to $S_e = \emptyset$ and $\mathrm{dom}(S_{\pi'}) \subseteq \mathrm{V}_{\delta^-}$, we have $x_1^{\delta^+} \notin \mathrm{dom}(R \cup S_e \cup S_{\pi'})$. Therefore, it suffices to show that $R \cup S_e \cup S_{\pi'}{\restriction}_{\mathrm{V}\backslash\{x_1^{\delta^+}\}}$ is well-founded. But this is well-founded as a subrelation of $R \cup S_e \cup S_\pi$, which is well-founded because π is $(e, \mathcal{S})$-compatible with (C, R). It remains to show that Item 2 of Definition 5.11 holds. By $x_1^{\delta^+} \notin \mathcal{V}(\langle \mathrm{V} \backslash \{x_1^{\delta^+}\}\rangle C)$, as π is $(e, \mathcal{S})$-compatible with (C, R), it suffices to show Item 2 only for $x_1^{\delta^+}$. Let $\tau : \mathrm{V}_{\delta^-} \to \mathcal{S}$ and $\eta : \{x_1^{\delta^+}\} \to \mathcal{S}$ be arbitrary. Set $\delta_0 := \epsilon(\pi)(\tau) \uplus \tau$, $\delta_1 := \epsilon(\pi')(\tau) \uplus \tau$, $\delta'_i := \eta \uplus_{\mathrm{V}\backslash\{x_1^{\delta^+}\}}{\upharpoonleft}\delta_i$, and $\delta'' := \{x_0^{\delta^+} \mapsto \eta(x_1^{\delta^+})\} \uplus_{\mathrm{V}\backslash\{x_0^{\delta^+}\}}{\upharpoonleft}\delta_0$. Suppose that $A_1\{x \mapsto x_1^{\delta^+}\}$ is $(\delta'_1, e, \mathcal{S})$-valid. We have to show the claim that $A_1\{x \mapsto x_1^{\delta^+}\}$ is $(\delta_1, e, \mathcal{S})$-valid. As $\delta'_0 = \delta'_1$, $A_1\{x \mapsto x_1^{\delta^+}\}$ is $(\delta'_0, e, \mathcal{S})$-valid. As π is $(e, \mathcal{S})$-compatible with (C, R), we have that $A_1\{x \mapsto x_1^{\delta^+}\}$ is $(\delta_0, e, \mathcal{S})$-valid. If $\forall x.\ (A_0 \Leftrightarrow A_1)$ is not $(\delta_0, e, \mathcal{S})$-valid, then $\delta_0 = \delta_1$, and the claim holds. Otherwise, as $x_1^{\delta^+} \notin \mathcal{V}(A_0, A_1)$ and ${}_{\mathrm{V}\backslash\{x_1^{\delta^+}\}}{\upharpoonleft}\delta'_1 = {}_{\mathrm{V}\backslash\{x_1^{\delta^+}\}}{\upharpoonleft}\delta_0 = {}_{\mathrm{V}\backslash\{x_1^{\delta^+}\}}{\upharpoonleft}\delta_1$, we know that $A_0\{x \mapsto x_1^{\delta^+}\}$ is $(\delta'_1, e, \mathcal{S})$-valid and it suffices to show that $A_0\{x \mapsto x_1^{\delta^+}\}$ is $(\delta_1, e, \mathcal{S})$-valid. By $x_i^{\delta^+} \notin \mathcal{V}(A_0, A_1)$, $A_0\{x \mapsto x_0^{\delta^+}\}$ is $(\delta'', e, \mathcal{S})$-valid and it suffices to show (note that $\epsilon(\pi')(\tau)(x_1^{\delta^+}) = \epsilon(\pi)(\tau)(x_0^{\delta^+})$) that $A_0\{x \mapsto x_0^{\delta^+}\}$ is $(\delta_0, e, \mathcal{S})$-valid, but this is the case indeed, because π is $(e, \mathcal{S})$-compatible with (C, R). **Q.e.d. (Lemma 5.18)**

As already noted in § 4.6, the single-formula sequent $Q_C(y^{\delta^+})$ of Definition 4.6 is a formulation of axiom (ε_0) of § 2.1.3 in our framework.

Lemma 5.19 ((C, R)-Validity of $Q_C(y^{\delta^+})$) *Let C be an R-choice-condition. Let $y^{\delta^+} \in \mathrm{dom}(C)$. The formula $Q_C(y^{\delta^+})$ is (C, R)-valid. Moreover, $Q_C(y^{\delta^+})$ is $(\pi, e, \mathcal{S})$-valid for any Σ-structure $\mathcal{S}$, any $(\mathcal{S}, R)$-valuation e, and any π that is $(e, \mathcal{S})$-compatible with (C, R).*

Proof of Lemma 5.19 Let $C(y^{\delta^+}) = \lambda v_0.\ \ldots \lambda v_{l-1}.\ B$ for a formula B. Then we have $Q_C(y^{\delta^+}) = \forall v_0.\ \ldots \forall v_{l-1}.\ (\ \exists y.\ (B\{y^{\delta^+}(v_0) \cdots (v_{l-1}) \mapsto y\})\ \Rightarrow\ B\ )$ for an arbitrary $y \in \mathrm{V}_{\mathrm{bound}} \backslash \mathcal{V}(C(y^{\delta^+}))$. For π being $(e, \mathcal{S})$-compatible with (C, R), the $(\pi, e, \mathcal{S})$-validity follows now directly from Item 2 of Definition 5.11, according to the short discussion following Definition 4.6. The rest is trivial. **Q.e.d. (Lemma 5.19)**

5.7 Reduction

Reduction is the reverse of consequence. It is the backbone of logical reasoning, especially of abduction and goal-directed deduction. Our version of reduction does not only reduce a set of problems to another set of problems but also guarantees that the solutions of the latter also solve the former; where "solutions" means the valuations for the rigid variables, i.e. for the free γ-variables and the free δ^+-variables.

Definition 5.20 (Reduction)
Let C be an R-choice-condition.
Let $\mathcal{S}$ be a Σ-structure, and let G_0 and G_1 be sets of sequents.
G_0 *(C,R)-reduces to G_1 in $\mathcal{S}$* if
for any $(\mathcal{S},R)$-valuation e and any π that is $(e,\mathcal{S})$-compatible with (C,R):
if G_1 is $(\pi,e,\mathcal{S})$-valid, then G_0 is $(\pi,e,\mathcal{S})$-valid.

Theorem 5.21 (Reduction)

Let C be an R-choice-condition; $\mathcal{S}$ a Σ-structure; G_0, G_1, G_2, and G_3 sets of sequents.

1. (Validity) *If G_0 (C,R)-reduces to G_1 in $\mathcal{S}$ and G_1 is (C,R)-valid in $\mathcal{S}$, then G_0 is (C,R)-valid in $\mathcal{S}$, too.*

2. (Reflexivity) *In case of $G_0 \subseteq G_1$: G_0 (C,R)-reduces to G_1 in $\mathcal{S}$.*

3. (Transitivity) *If G_0 (C,R)-reduces to G_1 in $\mathcal{S}$ and G_1 (C,R)-reduces to G_2 in $\mathcal{S}$, then G_0 (C,R)-reduces to G_2 in $\mathcal{S}$.*

4. (Additivity) *If G_0 (C,R)-reduces to G_2 in $\mathcal{S}$ and G_1 (C,R)-reduces to G_3 in $\mathcal{S}$, then $G_0 \cup G_1$ (C,R)-reduces to $G_2 \cup G_3$ in $\mathcal{S}$.*

5. (Monotonicity) *For (C',R') being an extension of (C,R):*

(a) If G_0 is (C',R')-valid in $\mathcal{S}$, then G_0 is (C,R)-valid in $\mathcal{S}$.

(b) If G_0 (C,R)-reduces to G_1 in $\mathcal{S}$, then G_0 (C',R')-reduces to G_1 in $\mathcal{S}$.

6. (Instantiation) *For an R-substitution σ on $\mathrm{V}_{\gamma\delta^+}$, the extended σ-update (C',R') of (C,R), and for $O := \mathrm{dom}(C) \cap \mathrm{dom}(\sigma) \cap R^*\langle \mathcal{V}_{\delta^+}(G_0,G_1)\rangle$:*

(a) If $G_0\sigma \cup (\langle O\rangle Q_C)\sigma$ is (C',R')-valid in $\mathcal{S}$, then G_0 is (C,R)-valid in $\mathcal{S}$.

(b) If G_0 (C,R)-reduces to G_1 in $\mathcal{S}$, then $G_0\sigma$ (C',R')-reduces to $G_1\sigma \cup (\langle O\rangle Q_C)\sigma$ in $\mathcal{S}$.

Proof of Theorem 5.21
Items 1 to 5 are the Items 1 to 5 of Lemma 2.31 of [Wirth, 2004].
Item 6 follows from Lemma B.6 of [Wirth, 2004] when we set the meta variable N of Lemma B.6 to $\mathrm{dom}(C) \cap \langle(\mathrm{dom}(C) \cap \mathrm{dom}(\sigma)) \setminus O\rangle R^*$. **Q.e.d. (Theorem 5.21)**

Items 1 to 5 of Theorem 5.21 are straightforward. Item 6 is only technically complicated. Roughly speaking, the idea behind Item 6 is that reduction stays invariant under global application of the substitution σ on rigid variables, provided that we change from (C,R) to its extended σ-update (C',R') and that, in case that σ replaces some free δ^+-variable y^{δ^+} constrained by the choice-condition C, we can establish that this is a proper choice by showing $(Q_C(y^{\delta^+}))\sigma$, cf. Definition 4.6. The rest of this § 5.7 will give further explanation on the application of Theorem 5.21 and especially of Item 6.

Example 5.22 (Instantiation with $\mathrm{dom}(C) \cap \mathrm{dom}(\sigma) = \emptyset$)
For a simple application of Theorem 5.21(6b), where no free δ^+-variables occur and only a free γ-variable is instantiated, let us have a glimpse at the example proof of [Wirth, 2004, § 3.3]. Let G_0 be the proposition we want to prove, namely $\{\ z_0^{\gamma}(x_0^{\delta^-})(y_0^{\delta^-}) \prec \mathsf{ack}(x_0^{\delta^-}, y_0^{\delta^-})\ \}$, which says that Ackermann's function has a lower bound that is to be determined during the proof. Moreover, let G_1 —together with variable-condition R and R-choice-condition $\emptyset$— represent the current state of the proof. Then G_0 $(\emptyset, R)$-reduces to G_1. Moreover, in the example, G_1 reduces to a known lemma when we apply the substitution $\sigma := \{z_0^{\gamma} \mapsto \lambda x.\, \lambda y.\, y\}$. Now, Theorem 5.21(6b) says that the instantiated (and $\lambda\beta$-reduced) theorem $\{\ y_0^{\delta^-} \prec \mathsf{ack}(x_0^{\delta^-}, y_0^{\delta^-})\ \}$ $(\emptyset, R)$-reduces to the instantiated proof state $G_1\sigma$ and thus is $(\emptyset, R)$-valid by Theorem 5.21(3,1). Note that in this case the extended σ-update of $(\emptyset, R)$ is $(\emptyset, R)$ itself, and we have $O = \emptyset$ due to $\mathrm{dom}(C) \cap \mathrm{dom}(\sigma) = \emptyset$. Moreover, by Theorem 5.21(6a), also the original $\{\ z_0^{\gamma}(x_0^{\delta^-})(y_0^{\delta^-}) \prec \mathsf{ack}(x_0^{\delta^-}, y_0^{\delta^-})\ \}$ is known to be $(\emptyset, R)$-valid, but who would be interested in this weaker result now?

Example 5.23 ((C, R)-Validity of (E2′)) *(continuing Lemma 5.18)*
Instead of the ugly semantical proof of (E2′) of Lemma 5.18 in § 5.6, let us give a formal proof of (E2′) in our framework on a very abstract level by applying Theorem 5.21. We will reduce the set containing the single-formula sequent of the formula (E2′) to a valid set. This will complete our proof by Item 1 of Theorem 5.21. In the following, be aware of the requirements on occurrence of the variables as described in Lemma 5.18. We extend (C, R) with a fresh variable y^{δ^+} with $C'(y^{\delta^+}) := \left(\begin{array}{rl} & (\forall x.\ (A_0 \Leftrightarrow A_1) \Rightarrow y^{\delta^+} = x_0^{\delta^+}) \\ \wedge & (\neg\forall x.\ (A_0 \Leftrightarrow A_1) \Rightarrow A_1\{x \mapsto y^{\delta^+}\}) \end{array}\right)$.
Of course, to satisfy Definition 5.9(1), the current variable-condition R must be extended to $R' := R \cup (\mathcal{V}_{\mathrm{free}}(A_0, A_1) \cup \{x_0^{\delta^+}\}) \times \{y^{\delta^+}\}$. Note that, if we had done this extension during the proof, we would have needed Item 5b to keep reduction invariant, but as there is no reduction sequence given yet, it suffices to use Item 5a instead. Similarly, instead of Item 6b, we apply Item 6a, with $\sigma := \{x_1^{\delta^+} \mapsto y^{\delta^+}\}$. Then we have $O \subseteq \mathrm{dom}(C) \cap \mathrm{dom}(\sigma) = \{x_1^{\delta^+}\}$. For (C'', R'') being the extended σ-update of (C', R'), Item 6a says that it suffices to show (C'', R'')-validity of the set with the two single-formula sequents $\forall x.\ (A_0 \Leftrightarrow A_1) \Rightarrow x_0^{\delta^+} = y^{\delta^+}$ and $(Q_{C'}(x_1^{\delta^+}))\sigma$. The latter sequent reads $(\exists x.\ A_1\{x \mapsto x_1^{\delta^+}\}\{x_1^{\delta^+} \mapsto x\} \Rightarrow A_1\{x \mapsto x_1^{\delta^+}\})\sigma$, i.e. $\exists x.\ A_1 \Rightarrow A_1\{x \mapsto y^{\delta^+}\}$. But a simple case analysis on $\forall x.\ (A_0 \Leftrightarrow A_1)$ shows that the whole set (C'', R'')-reduces to

$$\left\{\begin{array}{ll} \exists x.\ A_0 \ \Rightarrow\ A_0\{x \mapsto x_0^{\delta^+}\}; & \left(\begin{array}{c} \exists y.\left(\begin{array}{rl} & (\forall x.\ (A_0 \Leftrightarrow A_1) \Rightarrow y = x_0^{\delta^+}) \\ \wedge & (\neg\forall x.\ (A_0 \Leftrightarrow A_1) \Rightarrow A_1\{x \mapsto y\}) \end{array}\right) \\ \Rightarrow \left(\begin{array}{rl} & (\forall x.\ (A_0 \Leftrightarrow A_1) \Rightarrow y^{\delta^+} = x_0^{\delta^+}) \\ \wedge & (\neg\forall x.\ (A_0 \Leftrightarrow A_1) \Rightarrow A_1\{x \mapsto y^{\delta^+}\}) \end{array}\right) \end{array}\right) \end{array}\right\},$$

i.e. to $\{Q_{C''}(x_0^{\delta^+});\ \ Q_{C''}(y^{\delta^+})\}$, which is (C'', R'')-valid by Lemma 5.19. (Note that by Item 4 of Theorem 5.21 it would have been sufficient to show that each of the the formulas of the set (C'', R'')-reduces to some (C'', R'')-valid set.) Thus, $(\mathrm{E2}')\sigma$ is (C'', R'')-valid. By Item 6a this means that (E2′) is (C', R')-valid, and by Item 5a this means that (E2′) is (C, R)-valid, as was to be shown. Note that we have $R'' := R' \cup \{(y^{\delta^+}, x_1^{\delta^+})\}$, so that σ is an R'-substitution. Indeed, the graph of R'' is acyclic:

$$\begin{tikzcd}
x_0^{\delta^+} \arrow[r, "R'"'] & y^{\delta^+} \arrow[r, "R''"'] & x_1^{\delta^+} \\
\mathcal{V}_{\mathrm{free}}(A_0) \arrow[u, "R^+"'] \arrow[ur, "R'"'] & & \mathcal{V}_{\mathrm{free}}(A_1) \arrow[u, "R^+"'] \arrow[ul, "R'"']
\end{tikzcd}$$

(continuing Example 4.7)

Example 5.24 (Instantiation with Higher-Order Choice-Condition)
Suppose that

$$(\mathsf{p1})\quad \forall v.\ \big(\ \exists y.\ (v = y{+}1) \Rightarrow (v = \mathsf{p}(v){+}1)\ \big)$$

is one of our lemmas for the predecessor function p in the arithmetic of natural numbers, and that we want to use this lemma as justification for replacing $y^{\delta^+}_{(\mathsf{p1})}$ under R-choice-condition $C(y^{\delta^+}_{(\mathsf{p1})}) = \lambda v.\ (v = y^{\delta^+}_{(\mathsf{p1})}(v){+}1)$ globally with p. Note that this was required in Example 4.7. By Theorem 5.21(6), for $\sigma := \{y^{\delta^+}_{(\mathsf{p1})} {\mapsto} \mathsf{p}\}$, we have to show $(Q_C(y^{\delta^+}_{(\mathsf{p1})}))\sigma$, which does not seem to be any problem because $(Q_C(y^{\delta^+}_{(\mathsf{p1})}))\sigma$ is just the above lemma $(\mathsf{p1})$.

5.8 On the Design of Similar Operators

In § 1 we already mentioned that the semantic free-variable framework for our ε may serve as the paradigm for the design of other operators similar to our version of the ε. In this § 5.8, we give some general hints on the two screws which may be turned to achieve the intended properties of such new operators.

The one screw to turn is the definition of (C, R)-validity. For instance, the "some π" in Definition 5.17 is something we can play around with. Indeed, in [Wirth, 1998, Definition 5.7 (Definition 4.4 in short version)], we can read "any π" instead, which is just the opposite extreme; for which (E2′) of Lemma 5.18 is valid iff $\exists! x.\, A_0 \vee \exists x.\forall y.(x{=}y)$. In between of both extremes, we could design operators tailored for generalized quantifiers (e.g. with cardinality specifications) or for the special needs of specification and computation of semantics of discourses in natural language. Note that the changes of our general framework for these operators would be quite moderate: In any case, it is "any π" what we read in the important Lemma 5.19 and the crucial Definition 5.20. Roughly speaking, only Theorem 5.21(6a) for the case of $O \neq \emptyset$ as well as Theorem 5.21(5a) would become false for a different choice on the quantification of π in Definition 5.17. The reason why we prefer "some π" to "any π" here and in [Wirth, 2004] is that "some π" results in more valid formulas (e.g. (E2′)) and makes theorem proving easier. Contrary to "any π" and to all semantics in the literature, "some π" frees us from considering all possible choices: We just have to pick a single arbitrary one and fix it in a proof step. Moreover, "some π" is very close to Hilbert's intentions on ε-substitution as described best in [Hilbert & Bernays, 1968/70, Vol. II, § 2.4].

The other screw to turn is the definition of compatibility. For instance, by modifying Item 2 of Definition 5.11 we can strengthen the notion of compatibility in such a way that $\delta(y^{\delta^+})$ has to pick the *smallest* value such that B becomes $(\delta, e, \mathcal{S})$-valid. With that modification of compatibility it would be interesting to model the failed trials of Hilbert's group to show termination of ε-substitution in arithmetic before [Ackermann, 1940] as described in [Hilbert & Bernays, 1968/70, Vol. II, § 2.4].

All in all, in our conceptually disentangled framework for the ε, there are at least these two well-defined and conceptually simple screws to turn for a convenient adjustment to achieve similar operators for different purposes.

6 Examples and Discussion on Philosophy of Language

6.1 Motivation and Overview

In this § 6, we exemplify our version of Hilbert's ε with several linguistic standard examples. The reason for choosing philosophy of language and the semantics of sentences in natural language as the field for our examples is threefold:

($\aleph$) These examples are simple and easily comprehensible, even without linguistic expertise. Moreover, they provide interesting and relevant test cases for descriptive terms and their logical frameworks.

($\beth$) The choice of our examples is natural due to the close relation of our ε to semantics for indefinite (and definite) articles and anaphoric pronouns in some natural languages.

(We ignore, however, the generic, qualitative, metaphoric, and pragmatic effects of these indefinite determiners; cf. § 6.2.3.)

($\daleth$) We hope that linguists find our solutions to these standard examples interesting enough to evaluate our semantics on its usefulness for developing tools that may help to represent and compute the semantics of sentences and discourses in natural language.

(Although the careful reader will find some method in our preference for certain representations, it would go far beyond the scope of this paper to present concrete procedures for generating different representational variants and to decide on which of them to prefer.)

We will proceed as follows: In § 6.2 we introduce to the description of the semantics of determiners in natural languages, and show that the ε is useful for it. In § 6.3 we have a brief look at the linguistic literature on Hilbert's ε. In § 6.4 we discuss cases that are difficult to model with our ε, such as Henkin quantifiers and cyclic choice in Bach–Peters sentences. We look at problems with right-unique ε in § 6.5, at donkey sentences in § 6.6, and at the difficulty of capturing semantics of natural language with quantifiers in § 6.7.

To speed our hope expressed in ($\daleth$) above, we try to make this § 6 accessible without reading the formally involved previous § 5. Accordingly, we remind or inform the reader of the following: We apply Smullyan's classification (cf. [Smullyan, 1968]) of problem-reduction rules into α, β, γ, and δ, and call the quantifiers eliminated and the variables introduced by γ- and δ-steps, γ- and δ-*quantifiers* and *free* γ- and *free* δ-*variables*, respectively. Free γ-variables (written x^{γ}) are implicitly existentially quantified. Free δ^--variables (x^{δ^-}) are implicitly universally quantified. The structure of the quantification is represented in a variable-condition. A *variable-condition* is a directed acyclic graph on free variables. The value of a free variable may transitively depend on the predecessors in the variable-condition, with the exception of the free δ^--variables that may always take arbitrary values. Moreover, a free δ^+-variable such as x^{δ^+} is existentially quantified but must take a value that makes its choice-condition $C(x^{\delta^+})$ true—if such a choice is possible. In problem reduction, free δ^--variables behave as constant parameters, free γ-variables may be globally instantiated with any term that does not violate the current variable-condition, and the instantiation of free δ^+-variables must additionally satisfy the current choice-condition. Furthermore, a sequent is a list of formulas which denotes the disjunction of these formulas.

6.2 Introduction: Pro and Contra Reference

In this § 6.2, we introduce to the description of the semantics of determiners in natural languages, and show that the ε is useful for it. We take the historical path by shedding some light on the following two seminal philosophic papers, which capture most opposite views pro and contra reference in natural languages:

Pro: [Meinong, 1904a]: "Ueber Gegenstandstheorie" by Alexius Meinong (1853–1920)

Contra: [Russell, 1905a]: "On Denoting" by Bertrand Russell (1872–1970)

As our ε does not surrender to the present king of France, it could help Russell to return to his original appreciation of Meinong's ideas and position, which he still expressed in [Russell, 1905b].

6.2.1 I met a man. (R1)

In [Russell, 1905a], the affirmation (R1) is taken to be

$$\text{`I met } x \text{ and } x \text{ is human' is not always false.} \tag{R2}$$

(Russell's identification of being a man and being human is not relevant to us here.) From [Russell, 1919, Chapter XV], it becomes clear that (R2) means what today would be stated as

$$\exists x.\ \big(\ \mathsf{Met}(\mathsf{I}, x) \wedge \mathsf{Human}(x)\ \big) \tag{W1}$$

In our free-variable framework, we can omit the γ-quantifier of (W1) and replace its bound γ-variable x either with a free γ-variable x_1^γ or with a free δ^+-variable $x_0^{\delta^+}$ constrained with a tautological choice-condition. This results in one of the logically equivalent forms of either

$$\mathsf{Met}(\mathsf{I}, x_1^\gamma) \wedge \mathsf{Human}(x_1^\gamma) \tag{W2}$$

or

$$\mathsf{Met}(\mathsf{I}, x_0^{\delta^+}) \wedge \mathsf{Human}(x_0^{\delta^+}) \tag{W3}$$

with choice-condition

$$C(x_0^{\delta^+}) \ := \ \mathsf{true} \tag{W4}$$

Note that (W3)+(W4) is logically equivalent to each of (W1) and (W2) because we have chosen an implicit existential quantification for our free δ^+-variables. This could be changed to universal or generalized quantification for the design of operators for descriptive terms similar to our novel ε-operator. Our preferred reading of (R1), however, is the following:

$$\mathsf{Met}(\mathsf{I}, x_0^{\delta^+}) \tag{W5}$$

with choice-condition

$$C(x_0^{\delta^+}) \ := \ \mathsf{Met}(\mathsf{I}, x_0^{\delta^+}) \wedge \mathsf{Human}(x_0^{\delta^+}) \tag{W6}$$

Only if there is an emphasis on the conviction that it was a human indeed whom I met, i.e. that the choice-condition (W6) denotes, we would model (R1) as (W3)+(W6).

The logical equivalence of (W3)+(W6) with (W1) is nothing but a version of Hilbert's original axiom (ε_1) for elimination the existential quantifier in our free-variable framework, where $x_0^{\delta^+}$ replaces the ε-term $\varepsilon x.\ \big(\ \mathsf{Met}(\mathsf{I}, x) \wedge \mathsf{Human}(x)\ \big)$.

(W5)+(W6), however, is a proper logical consequence of (W1) in general: If (W1) is false, (W5)+(W6) still would be true if I were Redcap and met the wolf disguised as a human. A listener knowing for certain that there are no men in the forest may make sense out of what Redcap tells him by assuming that the man she met denotes a specific object, and then find out that it must be the wolf. Then (W5)+(W6) is true; to wit, pick the wolf for $x_0^{\delta^+}$. This is so because we may choose any object if the choice-condition does not denote, i.e. if it is unsatisfiable in the context under consideration.

6.2.2 Scott is the author of Waverley. (SW1)

(SW1) is another famous example from [Russell, 1919, Chapter XVI]. We model it as

$$\text{Scott} = z^{\delta^+} \tag{SW2}$$

with choice-condition

$$C(z^{\delta^+}) \ := \ \text{“}z^{\delta^+}\text{ is author of Waverley”} \tag{SW3}$$

For the choice-condition C of (SW3), our version $Q_C(z^{\delta^+})$ of Hilbert's original axiom (ε_0) reads

$$\exists z.\ \text{“}z\text{ is author of Waverley”} \ \Rightarrow\ \text{“}z^{\delta^+}\text{ is author of Waverley”}$$

For global application of the substitution $\sigma := \{z^{\delta^+} \mapsto \text{Scott}\}$, we have to show $(Q_C(z^{\delta^+}))\sigma$. This is valid, provided that Scott *is* author of Waverley. Thus, by Theorem 5.21(6a), we can infer the validity of (SW2)+(SW3) from the validity of the σ-instance of (SW2), which is the tautology "Scott = Scott". This is in blank opposition to the following statement ([Russell, 1905a]):

> "The proposition 'Scott was the author of Waverley'" ... "does not contain any constituent 'the author of Waverley' for which we could substitute 'Scott'."

One could try to defend Russell's statement by arguing that he may only have seen his reduced syntactical form

$$\exists z.\ \big(\ \forall y.\ \big(\ (y{=}z) \Leftrightarrow \text{“}y\text{ is the author of Waverley”}\ \big) \wedge \text{Scott} = z\ \big), \tag{SW4}$$

but—after a deeper contemplation on (SW4) and the concept of reference in general, and after a closer look at [Russell, 1905a] and [Russell, 1919]—Russell's statement seems to be just an outcome of Russell's strange philosophy of sometimes ignoring Kant's distinction on *a posteriori* and *a priori*, Frege's notion of *sense*, and, in general, the *functions of syntax and logical calculi, mixing them up with semantics*. That Russell did so becomes more obvious from the following simpler statement [Russell, 1919, p.175]:

> " 'Scott is Sir Walter' is the same trivial proposition as 'Scott is Scott'."

Beyond that, [Russell, 1905a] argues *contra* reference in general, but in favor of encoding reference into mere predicate logic as in (SW4). Advantages of (SW2)+(SW3) over (SW4), however, are its elegance and simplicity as well as the introduction of the formal reference

object z^{δ^+}, which is non-trivially constrained by (SW3) and may be reused for further reference.

6.2.3 Ignoring the Definite, Generic, Qualitative, Salient, Specific, &c.

As a most interesting standard example we will consider "a/the round quadrangle is quadrangular" in § 6.2.4. We do not want to emphasize "*the* round quadrangle" as we want to circumvent the extra complication introduced by the definite syntactical form, which may or may not indicate properties such as uniqueness, specifity, or salience. We also do not want to emphasize "*a* round quadrangle is quadrangular" as we are not interested in the generic reading

$$\forall x.\ \big(\ \mathsf{Round}(x) \wedge \mathsf{Quadrangular}(x) \quad \Rightarrow \quad \mathsf{Quadrangular}(x)\ \big).$$

To be precise, we have to specify that we are interested in *reference* (i.e. not in *predication*) and that the usage of indefinite determiners (articles and pronouns) we intend to mirror with the ε here is *particular* (i.e. not *generic*) and *referential* (i.e. not *qualitative*). Generic and qualitative usage of indefinite articles is *merely quantificational* and refers to a property (predication) and not to an object having it (reference). For example, in the sentence "An elephant is a huge animal." the "An" is generic and the "a" is qualitative. It simply says $\forall x.\ \big(\ \mathsf{Elephant}(x) \quad \Rightarrow \quad \mathsf{Huge}(x) \wedge \mathsf{Animal}(x)\ \big)$. Moreover, our modeling with the ε does not presuppose that the description *designates* or that it is *salient* or *specific*. Note that the distinctions on salience and specifity depend on a discourse and the referential status for the speaker, resp., which we do not take into account here.

6.2.4 The round quadrangle is quadrangular.

With the *proviso* of § 6.2.3, we now model "a/the round quadrangle is quadrangular" as

$$\mathsf{Quadrangular}(y_1^{\delta^+}) \tag{S}$$

with choice-condition

$$C(y_1^{\delta^+}) \quad := \quad \mathsf{Round}(y_1^{\delta^+}) \wedge \mathsf{Quadrangular}(y_1^{\delta^+}) \tag{C1}$$

The choice-condition $C(y_1^{\delta^+})$ of (C1) is equivalent to false if we have the sequent

$$\neg\mathsf{Round}(u^{\delta^-}),\ \ \neg\mathsf{Quadrangular}(u^{\delta^-}) \tag{A}$$

available as a lemma. Then, we may choose any object for $y_1^{\delta^+}$. If we chose a quadrangular one, (S) becomes true. Thus, the statement (S)+(C1) is valid, due to our choice of an implicit existential quantification for the free δ^+-variables. We cannot follow the critique of [Russell, 1905a] against [Meinong, 1904a] here, namely that y^{δ^+} would be "apt to infringe the Law of Contradiction". Firstly, y^{δ^+} is well-specified by (C1) and denoting a well-defined object. Secondly, we do not even follow [Russell, 1905a] insofar as undefined objects in a domain would be in conflict with the Law of Contradiction. Indeed, for very good practical reasons, we find

(H) linguistically motivated models with arbitrary *not really existing objects* in [Hobbs, 1996] and [Hobbs, 2003ff., § 2.2], (as part of the *ontological promiscuity* suggested for the *logical notation*, cf. [Hobbs, 2003ff., § 1, Note 14]), and

(W) logically motivated models with arbitrary *undefined objects* in [Kühler & Wirth, 1996], and [Wirth, 2009].

The "really existing" or "defined" objects are simply those for which a predicate "Rexists" or "Def" holds. This treatment obeys both the Law of Contradiction and the Law of the Excluded Middle. While in the approach of (H) we have to replace (A) with

$$\neg\mathsf{Round}(u^{\delta^-}),\ \ \neg\mathsf{Quadrangular}(u^{\delta^-}),\ \ \neg\mathsf{Rexists}(u^{\delta^-})$$

there is an additional possibility in the approach of (W) where the standard variables range over defined objects only: When the variable $y_1^{\delta^+}$ in (C1) is a *general* variable, ranging over the defined as well as the undefined objects, and the variable u^{δ^-} in (A) is a standard variable, ranging over the defined objects only, then there are models of (A) with undefined objects that are both round and quadrangular. Note that this syntactical trick of (W) is not just syntactical sugar improving the readability of formulas critically, but also cuts down logical inference by restricting notions such as matching, unification, and rewriting.

6.2.5 The round quadrangle is just as certainly round as it is quadrangular.

Russell's critique on [Meinong, 1904a] is justified, however, insofar as a single statement of [Meinong, 1904a] seems to be in conflict with the Law of Contradiction, indeed: Assuming (A) from above, there seems to be no way to model the 2nd line of the following sentence as true in two-valued logics [Meinong, 1904a, p. 8, modernized orthography]:

> „Nicht nur der vielberufene goldene Berg ist von Gold, sondern auch
> das runde Viereck ist so gewiss rund als es viereckig ist." (V)
>
> "Not only the notorious golden mountain is of gold, but also
> the round quadrangle is just as certainly round as it is quadrangular."
> (our translation)

Of course, one could consider a trivial generic reading:

$$\forall x.\ \big(\ \mathsf{Round}(x) \wedge \mathsf{Quadrangular}(x) \quad\Rightarrow\quad \mathsf{Round}(x) \wedge \mathsf{Quadrangular}(x)\ \big).$$

But this is most unlikely to be intended due to the definite forms of the articles, especially due to the one in the 1st line of (V). Our preferred reading of the 2nd line of (V) is

$$\mathsf{Round}(y_1^{\delta^+}) \wedge \mathsf{Quadrangular}(y_1^{\delta^+}), \tag{P}$$

referring to the choice-condition of (C1) above. Now (P) simplifies to false under the assumption of (A), unless either

- we restrict u^{δ^-} in (A) to the defined, as sketched in § 6.2.4 (for which there is no indication in [Meinong, 1904a], however), or else

- we choose some para-consistent logics, where a contradiction does not imply triviality, cf. [Priest & Tanaka, 2009] (for which, however, there is no indication in [Meinong, 1904a], either). Contrary to its promising title "Meinong's Theory of Objects and Hilbert's ε-symbol", [Costa &al., 1991] does not contain relevant information (neither on Meinong's Gegenstandstheorie nor on Hilbert's ε) besides sketching a para-consistent logic.

Alternatively, [Meinong, 1904a] may be consistently understood as follows:

Note that Meinong's "Gegenstand" is best translated as "object" and his "Gegenstandstheorie" as "Theory of Objects". Possible other translations would be "subject", "concept reference", or "referent with choice-condition", but neither "thing" nor "objective". According to [Meinong, 1904a, p. 40f.], Gegenstandstheorie is the most general aprioristic science, whereas metaphysics is the most general aposterioristic science. Thus, Gegenstandstheorie is not less general than metaphysics, and we should carefully exclude from Meinong's notion of a Gegenstand any connotation of being physical or realizable. There is no explicit definition of Gegenstand in [Meinong, 1904a], but only a parenthetical indication:

> ... „die Bezugnahme, ja das ausdrückliche Gerichtetsein auf jenes ‚etwas', oder wie man ja ganz ungezwungen sagt, auf einen Gegenstand" ... [Meinong, 1904a, p. 2]
>
> ... "the reference, indeed the explicit pointing to that 'something', or—as one would very informally say—to a Gegenstand" ... (our translation)
> ("pointing to" may be replaced with "aiming at")

Having clarified the notion of Gegenstand a little, let us come back to the 2nd line of (V). On the one hand, as stated already above, our preferred reading (P) contradicts the above sequent (A). On the other hand, the reading "$\mathsf{Round}(y_1^{\delta^+}) \Leftrightarrow \mathsf{Quadrangular}(y_1^{\delta^+})$" implies "$\neg\mathsf{Round}(y_1^{\delta^+}) \wedge \neg\mathsf{Quadrangular}(y_1^{\delta^+})$" by (A). This reading, however, is very unlikely to be intended by somebody whose German is as excellent as Meinong's. I assume that Meinong wanted to say that a specification for reference—such as $C(y_1^{\delta^+})$ in (C1) above—is meaningful and should denote, no matter whether there is an object that satisfies it. Indeed, we read:

> ... „was dem Gegenstande in keiner Weise äußerlich ist, vielmehr sein eigentliches Wesen ausmacht, in seinem Sosein besteht, das dem Gegenstand anhaftet, mag er sein oder nicht sein."
> [Meinong, 1904a, p.13]
>
> ... "what is not contingent to a Gegenstand but establishes its proper character constitutes its suchness, which sticks to the Gegenstand, may it be or not be."
> (our translation)

In this light, might the 2nd line of (V) even be boldly read as the *valid* statement

$$\mathsf{Round}(y_1^{\delta^+}) \wedge \mathsf{Quadrangular}(y_2^{\delta^+}), \tag{P$'$}$$

with (C1) and

$$C(y_2^{\delta^+}) \quad := \quad \mathsf{Round}(y_2^{\delta^+}) \wedge \mathsf{Quadrangular}(y_2^{\delta^+}) \tag{C2}$$

6.2.6 Conclusion

All in all, we may conclude that the ε—and especially our novel treatment of it—is useful for describing the semantics of determiners in natural languages: We can formalize some of Meinong's ideas on philosophy of language and contribute to the defense of his points of view against Russell's critique, even on empty descriptions.

6.3 A brief look at the Linguistic Literature on the ε

In this § 6.3, we have a brief look at the linguistically motivated literature on Hilbert's ε, which goes beyond our discussion in §§ 2 and 3. The usefulness of Hilbert's ε for the description of the semantics of natural language is simultaneously threatened by *right-uniqueness* and *uncommitted choice*, which seem to be opposite threats like Scylla and Charybdis, hard to pass by in between even for brave Ulysses.

Right-Uniqueness: A right-unique behavior of the ε is a problem in natural language. For example, the same phrase modeled as an ε-term does not necessarily denote the same object. Indeed, it may necessarily denote two different ones as in "If *a bishop* meets *a bishop*, ...".

Based on Natural Deduction (cf. [Gentzen, 1935], [Prawitz, 1965]), Wilfried P. M. Meyer-Viol presents in his PhD thesis [Meyer-Viol, 1995] most interesting results on the ε in intuitionistic logic and a lot of fascinating ideas on how to use it for computing the semantics of sentences in natural language. The latter ideas, however, suffer from a right-unique behavior of the ε. We will discuss more problems with the right-uniqueness requirement in § 6.5 along [Geurts, 2000].

Uncommitted Choice: A major advantage of reference in natural language is the possibility to refer to an object a second time. Thus, the ε can hardly be of any use in semantics of natural language without the possibility to express committed choice; cf. § 2.6. Note, however, that — to express committed choice — we need right-uniqueness unless we replace the ε-terms with free δ^+-variables; cf. §§ 2.6 and 3.1.6.

Already in 1993, Jan van Eijck addressed the double problem of Scylla and Charybdis in the first part of the following sentence:

> "What we want, instead, is to employ different choice functions as we go along, and to let the interpretation process fail in case no appropriate choice of φ is possible because there are no φs." [Eijck, 1993, p. 242f.]

The second part of this sentence, however, is a judgment contra the ε, which we cannot accept: If we want to model a natural language discourse, we have to introduce a reference object even if currently no salient object satisfies the choice-condition of its free δ^+-variable; moreover, even for the round quadrangle we have to introduce an object because we cannot talk about it otherwise.

Klaus von Heusinger seems to take the first part of Eijck's sentence as a task instead of a problem description: In Heusinger [1997], the right-uniqueness of the ε is kept, but the usefulness for describing the semantics of natural language is improved by adding a situational index to the ε-symbol that makes it possible to denote different choice functions explicitly; cf. (19a′) in our § 6.6 for an example. We will refer to this indexed ε as *"Heusinger's indexed ε-operator"*. It already occurs in the English draft paper [Heusinger, 1996]. The book [Heusinger, 1997], however, is a German monograph on applying Hilbert's epsilon to the semantics of noun phrases and pronouns in natural language, with a focus on salience. Heusinger's indexed ε-operator is used to describe the definite as well as

the indefinite article in specific as well as non-specific contexts, resulting in four different representations.[9] Cf. [Heusinger, 1997] for further reference on the ε in the semantics of natural language.

The possible advantage of our semantics for the ε is that it is not right-unique but admits commitment to choices. Thus, it may help brave Ulysses to avoid both threats.

6.4 Problematic Aspects of Our ε

In this § 6.4, we discuss some aspects whose modeling in our free variable framework with our ε may fail when we take the straightforward way. The reason for this partial failure is that the posed representational demands are in conflict with our requirement of well-foundedness or acyclicity on the variable-condition R of our R-choice-conditions, cf. §§ 4.2 and 5.2, Definition 5.9, and Example 5.10. These representational demands are *Henkin quantification* (§ 6.4.1) and *cyclic choice* in Bach–Peters sentences (§ 6.4.2). We also show how to overcome these two weaknesses in our framework by simple deviations, namely by *raising* and by *parallel choice*.

6.4.1 Henkin Quantification

In [Hintikka, 1974], quantifiers in first-order logic were found insufficient to give the precise semantics of some English sentences. In [Hintikka, 1996], *IF logic*, i.e. Independence-Friendly logic—a first-order logic with more flexible quantifiers—is presented to overcome this weakness. In [Hintikka, 1974], we find the following sentence:

> Some relative of each villager and some relative of each townsman
> hate each other. (H0)

Let us first change to a lovelier subject:

> Some loved one of each woman and some loved one of each man
> love each other. (H1)

For our purposes here, we consider (H1) to be equivalent to the following sentence, which may be easier to understand and more meaningful:

> Every woman would love someone and every man would love someone,
> such that these loved ones would love each other.

(H1) can be represented by the following Henkin-quantified IF-logic formula:

$$\forall x_0.\left(\begin{array}{ll} & \mathsf{Female}(x_0) \\ \Rightarrow & \exists y_1.\left(\begin{array}{ll} & \mathsf{Loves}(x_0,y_1) \\ \wedge & \forall y_0.\left(\begin{array}{ll} & \mathsf{Male}(y_0) \\ \Rightarrow & \exists x_1/x_0.\left(\begin{array}{ll} & \mathsf{Loves}(y_0,x_1) \\ \wedge & \mathsf{Loves}(y_1,x_1) \\ \wedge & \mathsf{Loves}(x_1,y_1) \end{array}\right) \end{array}\right) \end{array}\right) \end{array}\right) \qquad \text{(H2)}$$

Let us refer to the standard game-theoretic semantics for quantifiers (cf. e.g. [Hintikka, 1996]), which is defined as follows: Witnesses have to be picked for the quantified variables outside–in. We have to pick the witnesses for the γ-quantifiers (i.e. in (H2) for the existential quantifiers), and our opponent picks the witnesses for the δ-quantifiers (i.e. for the universal quantifiers in (H2)). We win iff the resulting quantifier-free formula evaluates to true. A formula is valid iff we have a winning strategy.

Then the Henkin quantifier "$\exists x_1/x_0.$" in (H2) is a special quantifier which is a little different from "$\exists x_1.$". Game-theoretically, the Henkin quantifier "$\exists x_1/x_0.$" has the following

semantics: It asks us to pick the loved one x_1 independently from the choice of the woman x_0 (by our opponent in the game), although the Henkin quantifier occurs in the scope of the quantifier "$\forall x_0.$".

An alternative way to define the semantics of Henkin quantifiers is by describing their effect on the logically equivalent *raised* forms of the formulas in which they occur. *Raising* is a dual of Skolemization, cf. [Miller, 1992]. The raised version is defined as usual, besides that γ-quantifiers say "$\exists x_1$" followed by a slash as in "$\exists x_1/x_0.$" are raised without x_0 appearing as an argument to the raising function for x_1.

According to this, (H2) is logically equivalent to its following raised form (H3), where x_0 does not occur as an argument to the raising function x_1, which would be the case if we had a usual γ-quantifier "$\exists x_1.$" instead of "$\exists x_1/x_0.$" in (H2).

$$\exists x_1, y_1.\ \forall x_0, y_0.\ \left(\begin{array}{ll} & \mathsf{Female}(x_0) \\ \Rightarrow & \left(\begin{array}{ll} & \mathsf{Loves}(x_0, y_1(x_0)) \\ \wedge & \left(\begin{array}{ll} & \mathsf{Male}(y_0) \\ \Rightarrow & \left(\begin{array}{ll} & \mathsf{Loves}(y_0, x_1(y_0)) \\ \wedge & \mathsf{Loves}(y_1(x_0), x_1(y_0)) \\ \wedge & \mathsf{Loves}(x_1(y_0), y_1(x_0)) \end{array}\right) \end{array}\right) \end{array}\right) \end{array}\right) \tag{H3}$$

Besides moving–out the γ-quantifiers from (H2) to (H3), we can also move–out the range restriction $\mathsf{Male}(y_0)$ of y_0, yielding the following, again logically equivalent form (H4), which nicely reflects the symmetry of (H1):

$$\exists x_1, y_1.\ \forall x_0, y_0.\ \left(\left(\begin{array}{ll} & \mathsf{Female}(x_0) \\ \wedge & \mathsf{Male}(y_0) \end{array}\right) \Rightarrow \left(\begin{array}{ll} & \mathsf{Loves}(x_0, y_1(x_0)) \\ \wedge & \mathsf{Loves}(y_0, x_1(y_0)) \\ \wedge & \mathsf{Loves}(y_1(x_0), x_1(y_0)) \\ \wedge & \mathsf{Loves}(x_1(y_0), y_1(x_0)) \end{array}\right)\right) \tag{H4}$$

Now, (H4) looks already very much like the following tentative representation of (H1) in our framework of free variables:

$$\left(\begin{array}{ll} & \mathsf{Female}(x_0^{\delta^-}) \\ \wedge & \mathsf{Male}(y_0^{\delta^-}) \end{array}\right) \Rightarrow \left(\begin{array}{ll} & \mathsf{Loves}(x_0^{\delta^-}, y_1^{\delta^+}) \\ \wedge & \mathsf{Loves}(y_0^{\delta^-}, x_1^{\delta^+}) \\ \wedge & \mathsf{Loves}(y_1^{\delta^+}, x_1^{\delta^+}) \\ \wedge & \mathsf{Loves}(x_1^{\delta^+}, y_1^{\delta^+}) \end{array}\right) \tag{H1$'$}$$

with choice-condition C given by
$$\begin{array}{rcl} C(y_1^{\delta^+}) & := & \mathsf{Female}(x_0^{\delta^-}) \Rightarrow \mathsf{Loves}(x_0^{\delta^-}, y_1^{\delta^+}) \\ C(x_1^{\delta^+}) & := & \mathsf{Male}(y_0^{\delta^-}) \Rightarrow \mathsf{Loves}(y_0^{\delta^-}, x_1^{\delta^+}) \end{array}$$

which requires the variable-condition to contain $R_C := \{(x_0^{\delta^-}, y_1^{\delta^+}), (y_0^{\delta^-}, x_1^{\delta^+})\}$ by Definition 5.9(1). Note that we can add $(y_1^{\delta^+}, y_0^{\delta^-})$ to our variable-condition R here to express that $y_1^{\delta^+}$ must not read $y_0^{\delta^-}$, which results in a logical equivalence to the original formula (H2) but with a standard γ-quantification "$\exists x_1.$" instead of the Henkin quantification "$\exists x_1/x_0.$".

If we tried to model the Henkin quantifier by adding $(x_1^{\delta^+}, x_0^{\delta^-})$ to R in addition, our choice-condition C would not be an R-choice-condition anymore by Definition 5.9 due to the following cycle:

$$\begin{tikzcd}
y_1^{\delta^+} \arrow[dr, "R"'] & x_0^{\delta^-} \arrow[l, "R_C"'] \\
x_1^{\delta^+} \arrow[ur, dotted, "R"] & y_0^{\delta^-} \arrow[l, "R_C"']
\end{tikzcd}$$

As shown in Example 2.9 of [Wirth, 2004], the δ^+-rules from § 4.2 become unsound when we admit such cycles. Without the δ^+-rules we could argue that R_C means something like "is read by" and that R means something like "must not read", so that it would be sufficient to require only the given irreflexivity of $R_C{\circ}R$, instead of the irreflexivity of the transitive closure of $R_C{\cup}R$, which is nothing but the acyclicity of $R_C{\cup}R$. Such "weak forms" are indeed sound for δ^--rules (cf. [Wirth, 2004, Note 9]), but the price of abandoning the δ^+-rules (esp. in a framework for Hilbert's ε) is ridiculously high in comparison to an increased order of some variables, such as of x_1 and y_1 in (H4).

Let us compare the failure of our approach to represent Henkin quantifiers without raising on the one hand, with the situation in [Heusinger, 1996, p. 85] (where (H0) has the label (25)) on the other hand. It may be interesting to see that it is well possible to model Henkin quantifiers with a right-unique version of Hilbert's ε, cf. [Heusinger, 1996], p. 85, (25c). After replacing both "hating" and "being a relative" with "loving", adding the fact that the loved ones are not chosen from empty sets of candidates (i.e. the presupposition that they exist), using free δ^--variables for the outermost universal bound variables, correcting a flaw,[10] and enhancing readability by introducing two more free δ^--variables $x_1^{\delta^-}$ and $y_1^{\delta^-}$, (25c) of [Heusinger, 1996] reads:

$$\left(\begin{array}{ll} & \mathsf{Female}(x_0^{\delta^-}) \\ \wedge & \mathsf{Male}(y_0^{\delta^-}) \\ \wedge & x_1^{\delta^-} = \varepsilon x_1.\, \mathsf{Loves}(y_0^{\delta^-}, x_1) \\ \wedge & y_1^{\delta^-} = \varepsilon y_1.\, \mathsf{Loves}(x_0^{\delta^-}, y_1) \end{array}\right) \Rightarrow \left(\begin{array}{ll} & \mathsf{Loves}(x_0^{\delta^-}, y_1^{\delta^-}) \\ \wedge & \mathsf{Loves}(y_0^{\delta^-}, x_1^{\delta^-}) \\ \wedge & \mathsf{Loves}(y_1^{\delta^-}, x_1^{\delta^-}) \\ \wedge & \mathsf{Loves}(x_1^{\delta^-}, y_1^{\delta^-}) \end{array}\right) \qquad \text{(H5)}$$

To model the Henkin quantifier correctly, an ε-term such as "$\varepsilon x_1.\, \mathsf{Loves}(y_0^{\delta^-}, x_1)$" in (H5) must not depend on $x_0^{\delta^-}$. This is contrary to $x_1^{\delta^+}$ in (H1′), whose value may well depend on that of $x_0^{\delta^-}$, unless $(x_1^{\delta^+}, x_0^{\delta^-})$ is included in R^+. To achieve this independence, it is not necessary that the ε gets an extensional semantics. It suffices that the semantics of the ε-term does not depend on anything not named in its formula, namely "$\mathsf{Loves}(y_0^{\delta^-}, x_1)$" in our case. On the one hand, any of the semantics of § 3.1 satisfies this independence, but—due to its right-uniqueness—is not suitable for describing the semantics of determiners in natural languages, cf. § 6.3, Item "Right-Uniqueness". On the other hand, Heusinger's indexed ε-operator, however, does not necessarily satisfy this independence, because it may get information on $x_0^{\delta^-}$ out of its situational index, cf. § 6.3, below Item "Uncommitted Choice", and § 6.5.

Thus, the inability of our framework to capture Henkin quantifiers without raising is also implicitly present in all other known approaches suitable for describing the semantics of determiners in natural languages.

Moreover, raising cannot be avoided in the presence of explicit ε-terms because these terms are an equivalent to raising already.

Furthermore, in natural language, Henkin quantification is typically ambiguous and the Henkin-quantified versions are always logically stronger than the ones with usual γ-quantifiers instead. Thus, it appears to be advantageous to have more flexibility in computing the semantics of sentences in natural language by starting with possibly weaker formulations such as (H1′). While we cannot represent the Henkin quantification in our framework without raising, we could start with the following raised version of (H1′).

$$\left(\begin{array}{ll} & \mathsf{Female}(x_0^{\delta^-}) \\ \wedge & \mathsf{Male}(y_0^{\delta^-}) \\ \wedge & x_1^{\delta^-} = x_3^{\delta^+}(x_0^{\delta^-})(y_0^{\delta^-}) \\ \wedge & y_1^{\delta^-} = y_2^{\delta^+}(x_0^{\delta^-}) \end{array}\right) \Rightarrow \left(\begin{array}{ll} & \mathsf{Loves}(x_0^{\delta^-}, y_1^{\delta^-}) \\ \wedge & \mathsf{Loves}(y_0^{\delta^-}, x_1^{\delta^-}) \\ \wedge & \mathsf{Loves}(y_1^{\delta^-}, x_1^{\delta^-}) \\ \wedge & \mathsf{Loves}(x_1^{\delta^-}, y_1^{\delta^-}) \end{array}\right) \tag{H2$'$}$$

with R-choice-condition C given by

$$\begin{array}{lll} C(y_2^{\delta^+}) & := & \lambda x.\ (\mathsf{Female}(x)\ \Rightarrow\ \mathsf{Loves}(x, y_2^{\delta^+}(x))) \\ C(x_2^{\delta^+}) & := & \lambda y.\ (\mathsf{Male}(y)\ \Rightarrow\ \mathsf{Loves}(y, x_2^{\delta^+}(y))) \\ C(x_3^{\delta^+}) & := & \lambda x.\ \lambda y.\ (\mathsf{Female}(x) \wedge \mathsf{Male}(y)\ \Rightarrow\ \mathsf{Loves}(y, x_3^{\delta^+}(x)(y))) \end{array}$$

which requires no extension of the variable-condition R. When we then find out that the sentence is actually meant to be Henkin quantified, we can apply the substitution $\sigma := \{x_3^{\delta^+} \mapsto \lambda u.\, x_2^{\delta^+}\}$. This turns (H2$'$) into a form equivalent to (H4), reflecting the intended semantics of (H1). Note that the condition $(Q_C(x_3^{\delta^+}))\sigma$ (cf. Definition 4.6), which is required for invariance of reduction under instantiation in Theorem 5.21(6), is

$$\forall x.\ \forall y.\ \left(\begin{array}{ll} & \exists z.\ (\mathsf{Female}(x) \wedge \mathsf{Male}(y)\ \Rightarrow\ \mathsf{Loves}(y, z)) \\ \Rightarrow & (\mathsf{Female}(x) \wedge \mathsf{Male}(y)\ \Rightarrow\ \mathsf{Loves}(y, (\lambda u.\, x_2^{\delta^+})(x)(y))) \end{array}\right)$$

and simplifies to

$$\forall y.\ \left(\begin{array}{ll} & \exists z.\ (\mathsf{Male}(y)\ \Rightarrow\ \mathsf{Loves}(y, z)) \\ \Rightarrow & (\mathsf{Male}(y)\ \Rightarrow\ \mathsf{Loves}(y, x_2^{\delta^+}(y))) \end{array}\right),$$

which is just $Q_C(x_2^{\delta^+})$, which is valid according to Lemma 5.19.

6.4.2 Cyclic Choices and Bach–Peters Sentences

As an example where references of an anaphor and a cataphor cross (i.e. a so-called "Bach–Peters sentence" after Emmon Bach and Stanley Peters), consider

A man who loves her marries a woman who, however, does not love him. (B0)

If we start with

$$\mathsf{Marries}(y_0^{\delta^+}, x_0^{\delta^+}) \tag{B1}$$

with R-choice-condition
$$\begin{array}{lll} C(y_0^{\delta^+}) & := & \mathsf{Male}(y_0^{\delta^+}) \wedge \mathsf{Loves}(y_0^{\delta^+}, f_1^{\gamma}(y_0^{\delta^+})) \wedge \mathsf{Female}(f_1^{\gamma}(y_0^{\delta^+})), \\ C(x_0^{\delta^+}) & := & \mathsf{Female}(x_0^{\delta^+}) \wedge \neg\mathsf{Loves}(x_0^{\delta^+}, f_2^{\gamma}(x_0^{\delta^+})) \wedge \mathsf{Male}(f_2^{\gamma}(x_0^{\delta^+})), \end{array}$$

then R^+ has to contain $\{(f_1^{\gamma}, y_0^{\delta^+}),\ (f_2^{\gamma}, x_0^{\delta^+})\}$ according to Definition 5.9(1). This says that the substitution $\sigma := \{f_1^{\gamma} \mapsto \lambda z.\, x_0^{\delta^+},\ f_2^{\gamma} \mapsto \lambda z.\, y_0^{\delta^+}\}$, which binds the pronouns "her" ($f_1^{\gamma}(y_0^{\delta^+})$) and "him" ($f_2^{\gamma}(x_0^{\delta^+})$) to their intended referents $x_0^{\delta^+}$ and $y_0^{\delta^+}$, resp., is not an R-substitution, however. This is due to the following cycle; cf. Definition 5.3:

$$\begin{tikzcd} y_0^{\delta^+} \arrow[drr, "R_\sigma"'] & & f_1^{\gamma} \arrow[ll, "R"'] \\ x_0^{\delta^+} \arrow[urr, "R_\sigma"] & & f_2^{\gamma} \arrow[ll, "R"] \end{tikzcd}$$

Indeed, the (extended) σ-updated (and $\lambda\beta$-reduced) choice-condition (C', R') of (C, R) (cf. Definition 5.15), namely
$$\begin{array}{lll} C'(y_0^{\delta^+}) & := & \mathsf{Male}(y_0^{\delta^+}) \wedge \mathsf{Loves}(y_0^{\delta^+}, x_0^{\delta^+}) \wedge \mathsf{Female}(x_0^{\delta^+}), \\ C'(x_0^{\delta^+}) & := & \mathsf{Female}(x_0^{\delta^+}) \wedge \neg\mathsf{Loves}(x_0^{\delta^+}, y_0^{\delta^+}) \wedge \mathsf{Male}(x_0^{\delta^+}), \end{array}$$
cannot be an R'-choice-condition for any (acyclic) variable-condition R', cf. Definition 5.9.

As we cannot choose $y_0^{\delta^+}$ before $x_0^{\delta^+}$ nor $x_0^{\delta^+}$ before $y_0^{\delta^+}$, we have to choose them in parallel. Thus, the only way to overcome this failure within our framework seems to be to start with

$$\mathsf{Marries}(1^{\mathrm{st}}(z^{\delta^+}), 2^{\mathrm{nd}}(z^{\delta^+})) \tag{B2}$$

with choice-condition

$$C_1(z^{\delta^+}) \quad := \quad \left(\begin{array}{rl} & \mathsf{Male}(1^{\mathrm{st}}(z^{\delta^+})) \wedge \mathsf{Loves}(1^{\mathrm{st}}(z^{\delta^+}), f_1^{\gamma}(z^{\delta^+})) \wedge \mathsf{Female}(f_1^{\gamma}(z^{\delta^+})) \\ \wedge & \mathsf{Female}(2^{\mathrm{nd}}(z^{\delta^+})) \wedge \neg\mathsf{Loves}(2^{\mathrm{nd}}(z^{\delta^+}), f_2^{\gamma}(z^{\delta^+})) \wedge \mathsf{Male}(f_2^{\gamma}(z^{\delta^+})) \end{array}\right)$$

where z^{δ^+} has the type of a pair and 1^{st} and 2^{nd} are its projections to the 1^{st} and 2^{nd} component, respectively. This requires the variable-condition R to contain $\{(f_1^{\gamma}, z^{\delta^+}),\ (f_2^{\gamma}, z^{\delta^+})\}$, which admits the substitution $\sigma' := \{f_1^{\gamma}{\mapsto}2^{\mathrm{nd}},\ f_2^{\gamma}{\mapsto}1^{\mathrm{st}}\}$ to be an R-substitution. Now, (B2) together with the (extended) σ'-update of C_1 (cf. Definition 5.15) captures the intended semantics of (B0) correctly.

Finally, note that a choice-condition of

$$C_2(z^{\delta^+}) \quad := \quad \left(\begin{array}{rl} & \mathsf{Male}(1^{\mathrm{st}}(z^{\delta^+})) \wedge \mathsf{Loves}(1^{\mathrm{st}}(z^{\delta^+}), f_1^{\gamma}(1^{\mathrm{st}}(z^{\delta^+}))) \wedge \mathsf{Female}(f_1^{\gamma}(1^{\mathrm{st}}(z^{\delta^+}))) \\ \wedge & \mathsf{Female}(2^{\mathrm{nd}}(z^{\delta^+})) \wedge \neg\mathsf{Loves}(2^{\mathrm{nd}}(z^{\delta^+}), f_2^{\gamma}(2^{\mathrm{nd}}(z^{\delta^+}))) \wedge \mathsf{Male}(f_2^{\gamma}(2^{\mathrm{nd}}(z^{\delta^+}))) \end{array}\right)$$

requires the substitution $\{f_1^{\gamma} \mapsto \lambda u.\,(2^{\mathrm{nd}}(z^{\delta^+})),\ f_2^{\gamma} \mapsto \lambda u.\,(1^{\mathrm{st}}(z^{\delta^+}))\}$, which is still no R-substitution because of the cycles between z^{δ^+} and f_i^{γ}. This means that—within cyclic choices—we should not restrict or project before all ambiguities have been resolved.

6.4.3 Conclusion

We have managed to overcome the two weaknesses of our framework exhibited in §§ 6.4.1 and 6.4.2 by simple deviations. For the Henkin quantifiers we had to increase the order of variables by *raising*. For the Bach–Peters sentences we had to replace a cycle of choices with a single *parallel choice*. As these problems are somehow unavoidable without paying high prices, this appears to be acceptable, especially because the partially ordered quantification required for natural languages in [Hintikka, 1974] is available for free in our framework of free variables of §§ 4.1 and 5.2.

Indeed, these inelegant aspects of our framework should not lead us to the conclusion to open Pandora's box by admitting cyclic choices. This would let most of the famous antinomies break into our system. If we admitted cyclic choices, we could not even say anymore whether a choice-condition can be satisfied for a certain free δ^+-variable or not. Example 5.10 in § 5.5 makes the essential problem obvious.

6.5 More Problems with a Right-Unique ε

In [Geurts, 2000], the use of Hilbert's ε in form of choice functions for the semantics of indefinites is attacked in several ways; and it is proposed that there is no way to interpret indefinites *in situ*, but that some form of "movement" is necessary, which, roughly speaking, may be interpreted as changing scopes of quantifiers. Although the examples given in [Geurts, 2000] are perfectly convincing in the given setting, we would like to point out that all the presented problems with the ε disappear when one uses a non-right-unique version such as ours. The following three example sentences and their labels are the ones of [Geurts, 2000].

6.5.1 All bicycles were stolen by a German. (1a)

We model this as

$$\mathsf{Bicycle}(x^{\delta^-}) \quad\Rightarrow\quad \mathsf{StolenBy}(x^{\delta^-}, y^{\delta^+})$$

with choice-condition

$$C(y^{\delta^+}) \quad:=\quad \mathsf{German}(y^{\delta^+})$$

If—in a first step—we find a model for this sentence with an empty variable-condition, then—in a second step—we can check whether it also satisfies a variable-condition that contains $(y^{\delta^+}, x^{\delta^-})$ in addition. A success of the first step provides us with a model for the weaker reading; a success of the second step with one for the stronger reading, too; i.e. that all bicycles were stolen by the same German. And this without "moving" any quantifiers or the like; which is, however, required when changing from

$$\forall x.\ \big(\ \mathsf{Bicycle}(x) \quad\Rightarrow\quad \exists y.\ (\mathsf{German}(y)\ \wedge\ \mathsf{StolenBy}(x,y))\ \big) \tag{1a-weak}$$

to

$$\exists y.\ \big(\ \mathsf{German}(y)\ \wedge\ \forall x.\ (\mathsf{Bicycle}(x) \quad\Rightarrow\quad \mathsf{StolenBy}(x,y))\ \big) \tag{1a-strong}$$

For a more interesting problem with right-unique ε, let us consider the following example.

6.5.2 Every girl gave a flower to a boy she fancied. (5)

Ignoring past tense, we model this as

$$\mathsf{Girl}(x^{\delta^-}) \quad\Rightarrow\quad \mathsf{Give}(x^{\delta^-}, z^{\delta^+}, y^{\delta^+})$$

with choice-condition

$$\begin{array}{lcl} C(y^{\delta^+}) & := & \mathsf{Girl}(x^{\delta^-}) \quad\Rightarrow\quad \mathsf{Boy}(y^{\delta^+}) \wedge \mathsf{Loves}(x^{\delta^-}, y^{\delta^+}) \\ C(z^{\delta^+}) & := & \mathsf{Flower}(z^{\delta^+}) \end{array}$$

As a choice function must pick the identical element from an identical extension, in [Geurts, 2000] there is a problem with two girls who love all boys, but give their flowers to two different ones. This problem does not appear in our modeling because our semantical relation (cf. Definition 5.5) does not depend on the common extension of their love, but only has to contain $(x^{\delta^-}, y^{\delta^+})$, which is in accordance with our variable-condition, which also has to contain $(x^{\delta^-}, y^{\delta^+})$ due to our above choice-condition for y^{δ^+}, cf. Definition 5.9.

The same problem of a common extension but a different choice object—but now in all possible worlds and intensions—of the following example is again no problem for us.

6.5.3 Every odd number is followed by an even number that is not equal to it. (7)

We model this as

$$\mathsf{Odd}(x^{\delta^-}) \quad\Rightarrow\quad x^{\delta^-}{+}1 = y^{\delta^+}$$

with choice-condition

$$C(y^{\delta^+}) \qquad := \qquad \mathsf{Odd}(x^{\delta^-}) \quad\Rightarrow\quad \mathsf{Even}(y^{\delta^+}) \wedge y^{\delta^+} \neq x^{\delta^-}$$

All in all, there was no real reason to "move" quantifiers or the like and the arguments of [Geurts, 2000] are not justified in the absence of a right-unique behavior of the ε. Moreover, the moving of the quantifiers as from (1a-weak) to (1a-strong) above is more complex and less intuitive than adding $(y^{\delta^+}, x^{\delta^-})$ to the current variable-condition.

6.6 Donkey Sentences and Heusinger's Indexed ε-Operator

6.6.1 If a man has a donkey, he beats it. (D)

The word "syntax" in the modern sense seems to have its first occurrence in the voluminous writings of Chrysippus of Soloi (Asia Minor) (3rd century B.C.), not the son of Pelops in the Oedipus mythos, but, of course, after Zeno of Citium and Cleanthes of Assos, the third leader of the Stoic school. So-called *Chrysippus sentences* and *donkey sentences* demonstrate the difficulties of interaction of indefinite noun phrases in a conditional ("a man", "a donkey") and anaphoric pronouns referring to them in the conclusion ("he", "it"). Cf. e.g. [Heusinger, 1997, § 7] for references on donkey and Chrysippus sentences. If semantics is represented with the help of quantification, donkey sentences reveal difficulties resulting from quantifiers and their scopes. Let us have a closer look at two examples.

6.6.2 If a man loves a woman, she loves him. (L0)

If we start by modeling this tentatively as

$$\begin{array}{rl} & \exists y_0.\ \big(\ \mathsf{Male}(y_0) \wedge \exists x_0.\ \big(\ \mathsf{Female}(x_0) \wedge \mathsf{Loves}(y_0, x_0)\ \big)\ \big) \\ \Rightarrow & \mathsf{Female}(x_1^\gamma) \wedge \mathsf{Loves}(x_1^\gamma, y_1^\gamma) \wedge \mathsf{Male}(y_1^\gamma) \end{array} \tag{L1}$$

we have no chance to resolve the reference of the pronouns "she" and "him" (x_1^γ and y_1^γ) before we get rid of the quantifiers. If we apply δ^--rules (cf. § 4.2) (besides α- and β-rules) we end up with the three sequents

$$\begin{array}{l} \neg\mathsf{Male}(y_0^{\delta^-}),\ \ \neg\mathsf{Female}(x_0^{\delta^-}),\ \ \neg\mathsf{Loves}(y_0^{\delta^-}, x_0^{\delta^-}),\ \ \mathsf{Female}(x_1^\gamma) \\ \neg\mathsf{Male}(y_0^{\delta^-}),\ \ \neg\mathsf{Female}(x_0^{\delta^-}),\ \ \neg\mathsf{Loves}(y_0^{\delta^-}, x_0^{\delta^-}),\ \ \mathsf{Loves}(x_1^\gamma, y_1^\gamma) \\ \neg\mathsf{Male}(y_0^{\delta^-}),\ \ \neg\mathsf{Female}(x_0^{\delta^-}),\ \ \neg\mathsf{Loves}(y_0^{\delta^-}, x_0^{\delta^-}),\ \ \mathsf{Male}(y_1^\gamma) \end{array} \tag{L2}$$

and a variable-condition R including $\{x_1^\gamma, y_1^\gamma\} \times \{x_0^{\delta^-}, y_0^{\delta^-}\}$, which says that the substitution

$$\sigma^- \ := \ \{x_1^\gamma \mapsto x_0^{\delta^-},\ \ y_1^\gamma \mapsto y_0^{\delta^-}\}$$

which turns the first and last sequents into tautologies and the middle one (L2) into the intended reading of (L0), is not an R-substitution and must not be applied, cf. Definition 5.3. Using δ^+-rules instead of the δ^--rules we get

$$\begin{array}{l} \neg\mathsf{Male}(y_0^{\delta^+}),\ \ \neg\mathsf{Female}(x_0^{\delta^+}),\ \ \neg\mathsf{Loves}(y_0^{\delta^+}, x_0^{\delta^+}),\ \ \mathsf{Female}(x_1^\gamma) \\ \neg\mathsf{Male}(y_0^{\delta^+}),\ \ \neg\mathsf{Female}(x_0^{\delta^+}),\ \ \neg\mathsf{Loves}(y_0^{\delta^+}, x_0^{\delta^+}),\ \ \mathsf{Loves}(x_1^\gamma, y_1^\gamma) \\ \neg\mathsf{Male}(y_0^{\delta^+}),\ \ \neg\mathsf{Female}(x_0^{\delta^+}),\ \ \neg\mathsf{Loves}(y_0^{\delta^+}, x_0^{\delta^+}),\ \ \mathsf{Male}(y_1^\gamma) \end{array}$$

and a variable-condition R including $\{(y_0^{\delta^+}, x_0^{\delta^+})\}$ instead. After application of the R-substitution $\sigma^+ \ := \ \{x_1^\gamma \mapsto x_0^{\delta^+},\ \ y_1^\gamma \mapsto y_0^{\delta^+}\}$, the instance of (L1) reduces to

$$\neg\mathsf{Male}(y_0^{\delta^+}),\ \ \neg\mathsf{Female}(x_0^{\delta^+}),\ \ \neg\mathsf{Loves}(y_0^{\delta^+}, x_0^{\delta^+}),\ \ \mathsf{Loves}(x_0^{\delta^+}, y_0^{\delta^+}) \tag{L3}$$

which is valid in a utopia where love is symmetric. A closer look reveals that our σ^+-updated variable-condition R now looks like $x_1^\gamma \longleftarrow x_0^{\delta^+} \longleftarrow y_0^{\delta^+} \longrightarrow y_1^\gamma$, while our ($\sigma^+$-updated) R-choice-condition C is

$$\begin{array}{rcl} C(y_0^{\delta^+}) & := & \mathsf{Male}(y_0^{\delta^+}) \ \ \wedge \ \ \exists x_0.\ \big(\ \mathsf{Female}(x_0) \wedge \mathsf{Loves}(y_0^{\delta^+}, x_0)\ \big) \\ C(x_0^{\delta^+}) & := & \mathsf{Female}(x_0^{\delta^+}) \wedge \mathsf{Loves}(y_0^{\delta^+}, x_0^{\delta^+}) \end{array}$$

But even if (L3) may be valid, this is not what we wanted to say in (L0), where "she" and "he" are obviously meant to be universal (strong, δ^-).

Thus, we had better start *without quantifiers from the very beginning*, namely directly with

$$\begin{array}{rl} & \mathsf{Male}(y_0^{\delta^-}) \wedge \mathsf{Loves}(y_0^{\delta^-}, x_0^{\delta^-}) \wedge \mathsf{Female}(x_0^{\delta^-}) \\ \Rightarrow & \mathsf{Female}(x_1^{\gamma}) \wedge \mathsf{Loves}(x_1^{\gamma}, y_1^{\gamma}) \wedge \mathsf{Male}(y_1^{\gamma}) \end{array} \tag{L4}$$

and empty variable-condition R', and then apply the R'-substitution σ^- from above to reduce its instance to

$$\neg\mathsf{Male}(y_0^{\delta^-}),\ \ \neg\mathsf{Loves}(y_0^{\delta^-}, x_0^{\delta^-}),\ \ \neg\mathsf{Female}(x_0^{\delta^-}),\ \ \mathsf{Loves}(x_0^{\delta^-}, y_0^{\delta^-}) \tag{L5}$$

which captures the universal meaning of (L0) properly.

Instead of a donkey sentence such as (L0) that prefers a genuinely universal reading as in (L5), the following donkey sentence prefers a partial switch to an existential reading:

6.6.3 If a bachelor loves a woman, he marries her. (M0)

If I love three utopian women, I am loved by all of them, but may marry at most one. Thus

$$\begin{array}{rl} & \mathsf{Male}(y_0^{\delta^-}) \wedge \mathsf{Loves}(y_0^{\delta^-}, x_0^{\delta^-}) \wedge \mathsf{Female}(x_0^{\delta^-}) \\ \Rightarrow & \mathsf{Male}(y_1^{\gamma}) \wedge \mathsf{Marries}(y_1^{\gamma}, x_1^{\gamma}) \wedge \mathsf{Female}(x_1^{\gamma}) \end{array} \tag{M1}$$

should be refined by application of $\{x_1^{\gamma} \mapsto x_1^{\delta^+},\ \ y_1^{\gamma} \mapsto y_0^{\delta^-}\}$ and simplification to

$$\neg\mathsf{Male}(y_0^{\delta^-}),\ \ \neg\mathsf{Loves}(y_0^{\delta^-}, x_0^{\delta^-}),\ \ \neg\mathsf{Female}(x_0^{\delta^-}),\ \ \mathsf{Female}(x_1^{\delta^+}) \tag{M2a}$$

$$\neg\mathsf{Male}(y_0^{\delta^-}),\ \ \neg\mathsf{Loves}(y_0^{\delta^-}, x_0^{\delta^-}),\ \ \neg\mathsf{Female}(x_0^{\delta^-}),\ \ \mathsf{Marries}(y_0^{\delta^-}, x_1^{\delta^+}) \tag{M2b}$$

with choice-condition

$$C(x_1^{\delta^+}) \quad := \quad \mathsf{Male}(y_0^{\delta^-}) \ \Rightarrow\ \mathsf{Loves}(y_0^{\delta^-}, x_1^{\delta^+}) \wedge \mathsf{Female}(x_1^{\delta^+}) \tag{C2}$$

On the one hand, if there is no women loved by the bachelor $y_0^{\delta^-}$, both (M2a) and (M2b) are valid. On the other hand, if there is at least one woman he loves, (M2a) is again valid (due to (C2)) and (M2b) expresses the intended reading of (M0).

Notice that we indeed have the possibility to let "woman" be universal (strong, δ^-) and "her" existential (weak, δ^+), picking one of the women loved by the bachelor—if there are any. Our elegant treatment is more flexible than a similar one of (D) along supposition theory in [Parsons, 1994]. Moreover, both these treatments are more lucid than the treatment of a sentence in [Heusinger, 1997], which is analogous to (M0): As (12) on p.183 of [Heusinger, 1997] we find the example

> „Wenn ein Mann einen Groschen hat, wirft er ihn in die Parkuhr."
>
> "If a man has a dime, he puts it into the meter." (our translation)

Mutandis mutatis and the readability improved, the modeling of (M0) according to (19a) on p.185 of [Heusinger, 1997] would be

$$\exists f.\ \forall i.\ \left(\begin{array}{rl} & \mathsf{Loves}(\ \varepsilon_i y.\, \mathsf{Male}(y),\ \ \varepsilon_{f(i)} x.\, \mathsf{Female}(x)\ ) \\ \Rightarrow & \mathsf{Marries}(\ \varepsilon_{a^*} y.\, \mathsf{Male}(y),\ \ \varepsilon_{a^*} x.\, \mathsf{Female}(x)\ ) \end{array} \right) \tag{19a$'$}$$

where the index a^* of Heusinger's indexed ε-operator (cf. § 6.3) seems to denote a choice function that chooses men as i does and women as $f(i)$ does. How a^* is to be formalized stays unclear in [Heusinger, 1997]. The real problem, however, is that (19a′) does not represent the intended meaning of (M0): To wit, take an f such that $f(i)$ always chooses a woman not loved by the man chosen by i; then (*ex falso quodlibet*) all our bachelors may stay unmarried, contradicting (M0). [11]

6.7 Quantifiers for Computing Semantics of Natural Language?

Representation of semantics of sentences and discourses in natural language with the help of quantifiers is of surprising difficulty. The examples in the previous §§ 6.5 and 6.6 indicate that quantified logic is problematic as a data structure for computing the semantics of sentences and discourses in natural language. Moreover, as already shown in § 6.4.1, for some sentences a precise representation with the quantifiers of first-order logic does not exist at all. Furthermore, the combinations of different scopes of quantifiers give rise to a combinatorial explosion of different readings: According to [Koller, 2004, p. 3], the following sentence, which is easy to understand for human beings, has "64764 different semantic readings, purely due to scope ambiguity", "even if one specified syntactic analysis" "is fixed":

> But that would give us all day Tuesday to be there.

I agree with [Hobbs, 1996] in that humans "do not compute the 120 possible readings" of

> In most democratic countries most politicians can fool most of the people on almost every issue most of the time.

Even if we can sometimes restrict the number of possible scopings below $n!$ for n quantifiers, e.g. by the algorithm of [Hobbs & Schieber, 1987], the number of possible readings is still too high for computers and human beings. Therefore, the relation of quantifiers to the semantics of natural language must be questioned. Notice that there are no quantifiers in natural language, and we can avoid them in the computation of their semantics with the help of the free-variable semantics introduced in this paper. Besides our most rudimental solution, we find the three following approaches to overcome quantifiers and scopes in the literature:

1. [Koller, 2004] uses standard quantified logic (plus bound variables outside the lexical scopes of their quantifiers) as basic language but leaves the formulas syntactically underspecified. A drawback seems to be that the actual formulas cannot be accessed.
2. [Hobbs, 1996] provides directly accessible formulas, namely some existentially quantified conjunctions. These formulas, however, are not likely to be close to the semantics of natural language as they are quite unreadable (to me at least). A modern modeling of the "typical elements" of [Hobbs, 1996] should be a new form of free δ-variables obtained by changing "some π" in Definition 5.17 into "each π" as in [Wirth, 1998, Definition 5.7 (Definition 4.4 in short version)], cf. our § 5.8. Two different "typical elements" of the same set (cf. [Hobbs, 1996, p. 6 of WWW version]) can then be modeled as two variables with the same choice-condition. Moreover, note that our use of reduction and instantiation in §§ 6.6.2 and 6.6.3 can be easily extended to a framework of *weighted abduction* as found in [Hobbs, 2003ff., Chapter 3].
3. Discourse Representation Theory (*DRT*, cf. e.g. [Kamp & Reyle, 1993], [Kamp &al., 2005]) shares with [Hobbs, 1996] the preference for existentially quantified conjunctions, but is not restricted to them. Nevertheless, the handling of quantifiers and scopes (or their substitutes) is quite impractical in DRT—even with the extensions for generalized quantifiers of [Kamp &al., 2005]. For example, DRT provides only one kind of free variables and no "typical elements", and universal quantification comes only with implications. Therefore, we expect an integration of our explicit characterization of free variables and our general way to introduce new tailored kinds of free variables into DRT to be beneficial. Note that also the accessibility restrictions of DRT can be captured by our variable-conditions, admitting more flexibility.

Gottlob Frege (1848–1925) invented first-order logic (including some second-order extension) in 1878 (so did Charles S. Peirce independently, cf. [Peirce, 1885]) and second-order logic including λ-abstraction and a ι-operator in 1893, both under the name "Begriffsschrift"; cf. [Frege, 1879; 1893/1903] and our Note 2, respectively. Frege designed his Begriffsschrift *not* for the task of computing the semantics of sentences in natural language, but actually—just as Guiseppe Peano (1858–1932) his ideography, cf. [Peano, 1896f.]—*to overcome the imprecision and ambiguity of natural language*. He cannot be blamed for the trouble quantifiers raise in representation and computation of the semantics of natural language. In [Frege, 1879], he is well aware of the difference of the semantics of natural language and his Begriffsschrift and compares it to that of the naked eye and the microscope. Indeed, Frege saw the Begriffsschrift as fundamentally different from natural language and as a substitute for it:

„Wenn es eine Aufgabe der Philosophie ist, die Herrschaft des Wortes über den menschlichen Geist zu brechen, indem sie die Täuschungen aufdeckt, die durch den Sprachgebrauch über die Beziehungen der Begriffe oft fast unvermeidlich entstehen, indem sie den Gedanken von demjenigen befreit, womit ihn allein die Beschaffenheit des sprachlichen Ausdrucksmittels behaftet, so wird meine Begriffsschrift, für diese Zwecke weiter ausgebildet, den Philosophen ein brauchbares Werkzeug werden können. Freilich gibt auch sie, wie es bei einem äußern Darstellungsmittel wohl nicht anders möglich ist, den Gedanken nicht rein wieder; aber einerseits kann man diese Abweichungen auf das Unvermeidliche und Unschädliche beschränken, andererseits ist schon dadurch, daß sie ganz anderer Art sind als die der Sprache eigentümlichen, ein Schutz gegen eine einseitige Beeinflussung durch eines dieser Ausdrucksmittel gegeben."

[Frege, 1879, p.VIf., modernized orthography]

"If it is a task of philosophy to break the dominance of natural language over the human mind

- by discovering the deceptions on the relations of notions resulting from the use of language often almost inevitably,
- by liberating the idea of what spoils it just by the linguistic means of expression,

then my Begriffsschrift—once further improved for these aims—will become a useful tool for the philosophers. Of course—as it seems to be unavoidable for any external means of representation—also the Begriffsschrift is not able to represent the idea undistortedly; but, on the one hand,

- it is possible to limit these distortions to the unavoidable and harmless, and, on the other hand
- a protection against a one-sided influence of one of these means of expression is given already because those of the Begriffsschrift are completely different from those characteristic of language." (our translation)

6.8 Conclusion

In this § 6, we have demonstrated our new indefinite semantics for Hilbert's ε and our free-variable framework in a series of interesting applications provided by standard examples from linguistics. *Can this serve as a paradigm useful in the specification and computation of semantics of discourses in natural language?* An investigation of this question requires a close collaboration of experts from both linguistics and logics. Be the answer to this question as it may, the field has provided us with an excellent test bed for descriptive terms and their logical frameworks.

7 Conclusion

Our novel indefinite semantics for Hilbert's ε presented in this paper was developed to solve the difficult soundness problems arising during the combination of mathematical induction in the liberal style of Fermat's *descente infinie* with state-of-the-art deduction.[12] Thereby, it had passed an evaluation of its usefulness even before it was recognized as a candidate for the semantics that David Hilbert probably had in mind for his ε. While the speculation on this question will go on, the semantical framework for Hilbert's ε proposed in this paper definitely has the following advantages:

Syntax: The requirement of a commitment to a choice is expressed syntactically and most clearly by the sharing of a free δ^+-variable, cf. § 4.5.

Semantics: The semantics of the ε is simple and straightforward in the sense that the ε-operator becomes similar to the referential use of the indefinite article in some natural languages. As we have seen in § 6, it is indeed so natural that it provides some help in understanding ideas on philosophy of language which were not easily accessible before. Our semantics for the ε is based on an abstract formal approach that extends a semantics for closed formulas (satisfying only very weak requirements, cf. § 5.3) to a semantics with several kinds of free variables: existential (γ), universal (δ^-), and ε-constrained (δ^+).

Reasoning: In a reductive proof step, our representation of an ε-term $\varepsilon x.\, A$ can be replaced with *any* term t that satisfies the formula $\exists x.\, A \;\Rightarrow\; A\{x{\mapsto}t\}$, cf. § 4.6. Thus, the soundness of such a replacement is likely to be expressible and verifiable in the original calculus. Our free-variable framework for the ε is especially convenient for developing proofs in the style of a working mathematician, cf. [Wirth, 2004; 2006]. Indeed, our approach makes proof work most simple because we do not have to consider all proper choices t for x (as in all other semantical approaches) but only a single arbitrary one, which is fixed in a proof step, just as choices are settled in program steps, cf. § 4.4.

Finally, we hope that new semantical framework will help to solve further practical and theoretical problems with the ε and improve the applicability of the ε as a logical tool for description and reasoning. Although we have only touched the surface of the subject in § 5.8, a tailoring of operators similar to our ε to meet the special demands of specification and computation in various areas (such as semantics of discourses in natural language) seems to be especially promising.

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Notes

Note 1 (History of the Symbols used to denote the ι-Binder)

It may be necessary to say something on the symbols used for the ι in the 19th and 20th century. In [Peano, 1896f.], Guiseppe Peano (1858–1932) wrote $\bar{\iota}$ instead of the ι of Example 2.1, and $\bar{\iota}\{\ x \mid A\ \}$ instead of $\iota x.\, A$. (Note that we have changed the class notation to modern standard here. We will do so in the following without mentioning it. Peano actually wrote $\overline{x\in}A$ instead of $\{\ x \mid A\ \}$ in [Peano, 1896f.].)

More than in Frege's logic calculus, Peano was interested in logic as a written language (ideography) with a clear description of its semantics in natural language. He also created an artificial substitute for natural language (*Latino sine flexione*, cf. e.g. [Kennedy, 2002]). Therefore, it does not come as a surprise that it was Peano who invented the *ι-binder*. Cf., however, Note 2 on Frege's ι-operator of 1893. In [Peano, 1899b], we find an alternative notation besides $\bar{\iota}$, namely a ι-symbol upside-down, i.e. inverted, i.e. rotated by π around its center. I do not know whether this is the first occurrence of the inverted ι-symbol. It was later used also in [Whitehead & Russell, 1910–1913], the infamous *Principia Mathematica* first published in 1910ff.. Thus, we should speak of *Peano's ι-symbol* and not of *Russell's ι-symbol*.

We call the famous *Principia Mathematica* infamous, because it is still rare and unaffordable, and—as standard notions and notation have changed quite a bit in the meanwhile—has become also quite incomprehensible for the occasional reader. It is a shame that there is no public interactive WWW version of the *Principia*, which facilitates look-up by translation into modern notation and online help with obsolete names.

Let us come back to Peano's ι. The bar above as well as the inversion of the ι were to indicated that $\bar{\iota}$ was implicitly defined as the inverse operator of the operator ι defined by $\iota y := \{y\}$, which occurred already in [Peano, 1890] and still in [Quine, 1981].

The definition of $\bar{\iota}$ reads literally [Peano, 1896f., Definition 22]:

$$a \in K\ .\ \exists a : x, y \in a\ .\ \supset_{x,y}\ .\ x = y : \supset : x = \bar{\iota}a\ .=.\ a = \iota x$$

This straightforwardly translates into more modern notation as follows:

$$\text{For any class } a: \quad a \neq \emptyset \wedge \forall x, y.\ (x, y \in a \quad \Rightarrow \quad x = y) \quad \Rightarrow \quad \forall x.\ (x = \bar{\iota}a \quad \Leftrightarrow \quad a = \iota x)$$

Giving up the flavor of an explicit definition of " $x = \bar{\iota}a$ ", this can be simplified to the following logically equivalent form:

$$\text{For any class } a: \quad \exists! x.\ x \in a \quad \Rightarrow \quad \bar{\iota}a \in a \tag{$\bar{\iota}_0$}$$

Besides notational difference, this is (ι_0) of our § 2.1.2.

It has become standard to write a simple non-inverted ι for the upside-down ι because Peano's original notation " ιy " has long ago been replaced with " $\{y\}$ " and because the upside-down ι is not easily available in today's typesetting. For instance, there does not seem to exist a TeX macro for it and—to enable font-independent archiving and republishing—some publishers do not permit the usage of nonstandard symbols.

Note 2 (Other ι-Operators Besides those of Russell, Hilbert, and Peano)

In [Frege, 1893/1903, Vol. I, § 11], we find another ι-operator. As this Vol. I was published by Gottlob Frege (1848–1925) in 1893, this seems to be the first occurrence of a ι-operator in the literature. The symbol he uses for the ι is a boldface backslash. As a *boldface* version of the backslash does not seem to be available in standard TEX, we use a simple backslash (\) here. Frege defines $\backslash\xi := x$ if there is some x such that $\forall y.\ (\xi(y) = (x = y))$. Writing the binder as a modern λ instead of Frege's *spiritus lenis*, Frege actually requires extensional equality of ξ and $\lambda y.\,(x = y)$. Now this would be basically Peano's ι-operator (cf. our § 2.1.2 and Note 1) unless Frege overspecified it by defining $\backslash\xi := \xi$ for all other cases.

Similarly, in set theories without urelements, the ι-operator is often defined by something like $\iota y.\,A := \{\ z \mid \exists x.\ (z \in x \wedge \forall y.\ (A \Leftrightarrow (x = y)))\ \}$ for new x and z, cf. e.g. [Quine, 1981]. This is again an overspecification resulting in $\iota y.\,A = \emptyset$ in case of $\neg\exists! y.\,A$.

Note 3 To be precise, in the standard predicate calculus of [Hilbert & Bernays, 1968/70] there are no axiom schemes but only axioms with predicate variables. The axiom schemes we use here simplify the presentation and refer to the *modified form of the predicate calculus* of [Hilbert & Bernays, 1968/70, Vol. II, p. 403], which is closer to today's standard syntax of first-order logic.

Note 4 (Consequences of the ε-Formula in Intuitionistic Logic)

Adding the ε either with (ε_0), with (ε_1), or with the ε-formula (cf. §§ 2.1.3 and 2.3) to intuitionistic first-order logic is equivalent on the ε-free theory to adding *Plato's Principle*, i.e. $\exists x.\ (\exists y.\,A \Rightarrow A\{y \mapsto x\})$ with x not occurring in A, cf. [Meyer-Viol, 1995, § 3.3].
Moreover, the non-trivial direction of (ε_2) is $\quad \forall x.\,A \quad \Leftarrow \quad A\{x \mapsto \varepsilon x.\,\neg A\}$.
Even intuitionistically, this entails its contrapositive $\quad \neg\forall x.\,A \quad \Rightarrow \quad \neg A\{x \mapsto \varepsilon x.\,\neg A\}$
and then, e.g. by the trivial direction of (ε_1) (when A is replaced with $\neg A$)

$$\neg\forall x.\,A \quad \Rightarrow \quad \exists x.\,\neg A \tag{Q2}$$

which is not valid in intuitionistic logic in general. Thus, the universal quantifier in Hilbert's intended object logic—if it includes (ε_2) or anything similar for the universal quantifier (such as Hilbert's τ-operator, cf. [Hilbert, 1923a])—is strictly weaker than in intuitionistic logic. More precisely, adding

$$\forall x.\,A \quad \Leftarrow \quad A\{x \mapsto \tau x.\,A\} \tag{τ_0}$$

is equivalent on the τ-free theory to adding $\exists x.\ (\forall y.\,A \Leftarrow A\{y \mapsto x\})$ with x not occurring in A, which again implies (Q2), cf. [Meyer-Viol, 1995, § 3.4.2].

From a semantical view, cf. [Gabbay, 1981], the intuitionistic $\forall$ may be eliminated, however, by first applying the Gödel translation into the modal logic S4 with classical $\forall$ and $\neg$, cf. e.g. [Fitting, 1999], and then adding the ε conservatively, e.g. by avoiding substitutions via λ-abstraction as in [Fitting, 1975].

Note 5 Besides the already mentioned extensional treatment of ε, in [Giese & Ahrendt, 1999] we also find an *intentional* treatment (which, roughly speaking, results from requiring the axiom (ε_0)) and a *substitutive* treatment where also the validity of the Substitution [Value] Lemma for ε-terms is required:

$$\mathrm{eval}\big(\ \mathcal{S}\ \big)\big(\ (\varepsilon x.\ A)\{y^{\mathrm{free}}{\mapsto}t\}\ \big) = \mathrm{eval}\big(\ \mathcal{S} \uplus \{y^{\mathrm{free}} \mapsto \mathrm{eval}(\mathcal{S})(t)\}\ \big)\big(\ \varepsilon x.\ A\ \big)$$

Here x is a bound and y^{free} is a free variable. Since logics where the Substitution Lemma for ε-free formulas does not hold are not considered (such as the first-order modal logic of [Fitting, 1999]), in [Giese & Ahrendt, 1999] we find a theorem basically saying that every extensional structure is substitutive.

Note 6
($0\neq 1,\ \ \varepsilon x.\,A_0 \neq \varepsilon x.\,A_1\ \ \Rightarrow\ \ \neg(\forall x.\,A_0\ \wedge\ \forall x.\,A_1)\ \ \vdash\ \ B \vee \neg B$ in intuitionistic logic)

For the proof of the weaker $0\neq 1$, (E2) $\vdash\ B \vee \neg B$ for any formula B, cf. already [Bell &al., 2001, Proof of Theorem 6.4], which already occurs in more detail in [Bell, 1993a, § 3], and is sketched in [Bell, 1993b, § 7].

Let B be an arbitrary formula. We are going to show that $\vdash B \vee \neg B$ holds in intuitionistic logic under the assumptions of reflexivity, symmetry, and transitivity of "=", the ε-formula (or (ε_0)), and of the formulas $0\neq 1$ and $\varepsilon x.\,A_0 \neq \varepsilon x.\,A_1\ \Rightarrow\ \neg(\forall x.\,A_0\ \wedge\ \forall x.\,A_1)$.

Let x be a variable not occurring in B. Set $A_i := (B \vee x = i)$.

Now what we have to show is a trivial consequence of the following Claims 1 and 2, $\vdash \varepsilon x.\,A_0 \neq \varepsilon x.\,A_1\ \Rightarrow\ \neg(\forall x.\,A_0\ \wedge\ \forall x.\,A_1)$, and Claim 3.

Claim 1: $0=0,\ 1=1,\ (\varepsilon\text{-formula})\{A{\mapsto}A_0, t{\mapsto}0\},\ (\varepsilon\text{-formula})\{A{\mapsto}A_1, t{\mapsto}1\}$
$\vdash\ B \vee (\varepsilon x.\,A_0 = 0\ \wedge\ \varepsilon x.\,A_1 = 1)$.

Claim 2: $\varepsilon x.\,A_0 = 0\ \wedge\ \varepsilon x.\,A_1 = 1,\ 0\neq 1,\ \forall x,y,z.\ (y = x\ \wedge\ y = z \Rightarrow x = z)$
$\vdash\ \varepsilon x.\,A_0 \neq \varepsilon x.\,A_1$.

Claim 3: $\neg(\forall x.\,A_0\ \wedge\ \forall x.\,A_1)\ \ \vdash\ \ \neg B$.

Proof of Claim 1: From the ε-formula and reflexivity of "=", we get $\vdash A_i\{x{\mapsto}\varepsilon x.\,A_i\}$. Thus, $\vdash A_0\{x{\mapsto}\varepsilon x.\,A_0\}\ \wedge\ A_1\{x{\mapsto}\varepsilon x.\,A_1\}$. From this, Claim 1 follows by distributivity.
Q.e.d. (Claim 1)

Proof of Claim 2: Trivial. Q.e.d. (Claim 2)

Proof of Claim 3: As x does not occur in B, we get $B \vdash \forall x.\,A_i$. The rest is trivial.
Q.e.d. (Claim 3)

Note 7 Regarding the classification of one of the δ-rules as "liberalized", we could try to object that $\mathcal{V}_{\mathrm{free}}(A)$ is not necessarily a subset of $\mathcal{V}_{\gamma\delta^+}(\Gamma\ \forall x.A\ \Pi)$, because it may include some additional free δ^--variables.

But the additional free δ^--variables blocked by the δ^+-rules (as compared to the δ^--rules) do not block proofs in practice. This has following reason: With a reasonably minimal variable-condition R, the only additional cycles that could occur are of the form $y^{\gamma\delta^+}\ R\ z^{\delta^-}\ R\ x^{\delta^+}\ R^+\ y^{\gamma\delta^+}$ with $y^{\gamma\delta^+}, z^{\delta^-} \in \mathcal{V}(\Gamma\ \forall x.A\ \Pi)$; unless we substitute something for x^{δ^+}. And in this case the corresponding δ^--rule would result in the cycle $y^{\gamma\delta^+}\ R\ x^{\delta^-}\ R^+\ y^{\gamma\delta^+}$ anyway.

Moreover, δ^--rules and free δ^--variables do not occur in inference systems with δ^+-rules before [Wirth, 2004], so that in the earlier systems $\mathcal{V}_{\mathrm{free}}(A)$ is indeed a subset of $\mathcal{V}_{\gamma\delta^+}(\Gamma\ \forall x.A\ \Pi)$.

Note 8 If the occurrences of y^{δ^+} in $C(y^{\delta^+})$ could differ in their arguments, there could be irresolvable conflicts on special arguments. And, in these conflicts, the choice of a *function as a whole* would essentially violate Hilbert's axiomatizations: As only terms and no functions are considered in [Hilbert & Bernays, 1968/70], the axiom schemes (ε_0) and (ε-*formula*) (cf. §§ 2.1.3 and 2.3) seem to require us to choose the values of this function individually. For example, in case of

$$C(y^{\delta^+}) \quad = \quad \lambda b.\ \big(\ y^{\delta^+}(b) \wedge \neg(y^{\delta^+}(\mathsf{true}) \wedge y^{\delta^+}(\mathsf{false}))\ \big),$$

for choosing y^{δ^+}, we are in conflict between $\lambda b'.\ (b' = \mathsf{false})$ (i.e. $\lambda b'.\neg b'$, for $C(y^{\delta^+})(\mathsf{false})$ to be true) and $\lambda b'.\ (b' = \mathsf{true})$ (i.e. $\lambda b'.\, b'$, for $C(y^{\delta^+})(\mathsf{true})$ to be true).

Note 9 (Do Salience, Specifity, and Uniqueness Determine Definiteness?)

Salience is the property of being known and prominent in discourse. *Specifity* is a property concerning the referential status for a speaker, expressing that he has a specific object in mind. Salience, specifity, and uniqueness are important aspects immanent in the distinction of definite and indefinite forms. In [Heusinger, 1997, p.1], we find the thesis that definiteness of articles expresses salience. This thesis is opposed to others emphasizing the aspects of *uniqueness* (as in the tradition of [Russell, 1905a]) or specifity instead of salience. The thesis is supported by the following two examples:

(definite, salient, specific, but not unique)

"The dog got in a fight with another dog."

[Heusinger, 1997, p. 20; our underlining]

(indefinite, not salient, but specific)

„Ich suche ein Buch, das ich gestern bekommen habe; es ist ein schönes." [Heusinger, 1997, p.16]

"I am looking for a book *which I got yesterday*; it is a beautiful one."

(our translation, our underlining)

Nevertheless, indefiniteness is typically unspecific:

(indefinite, not salient, unspecific)

„Ich suche (irgend) ein Buch; es soll ein schönes sein." [Heusinger, 1997, p.16]

"I am looking for *a*(n arbitrary) *book*; it is to be a beautiful one."

(our translation, our underlining)

Heusinger's thesis is not consistent, however, with the following example: Thomas Mann (1875–1955) starts his narration "Der kleine Herr Friedemann" as follows:

(definite, specific, but not salient)

„Die Amme hatte die Schuld. – " [Mann, 1898]

"The nurse bore the blame. — " (our translation, our underlining)

Obviously, none of uniqueness, salience, or specifity alone determines definiteness of articles: For uniqueness this becomes obvious from the first example already. For salience and specifity the following table may be helpful:

	SALIENT	NOT SALIENT
SPECIFIC	{The dog, ...}	{*a book*, ...} $\uplus$ {The nurse, ...}
UNSPECIFIC	$\emptyset$	{*a*(n arbitrary) *book*, ...}

If—as I conjecture—examples for "indefinite, but salient" do not exist, salience indeed requires definite forms; but not vice versa. In the technical treatment of salience with Heusinger's indexed ε-operator in [Heusinger, 1997], however, salience and definite forms indeed require each other.

Note 10 Note that our modeling of (H1) as (H4) of § 6.4.1 is correct, whereas the modeling of (H0) as (25c) of [Heusinger, 1996] is flawed: *Mutatis mutandis*, both "hating" and "being a relative" replaced with "loving", already the less complex (25b) of [Heusinger, 1996] has this flaw and reads:

$$\exists x_1, y_1.\ \forall x_0, y_0.\ \left(\left(\begin{array}{ll} & \mathsf{Female}(x_0) \\ \wedge & \mathsf{Male}(y_0) \\ \wedge & \mathsf{Loves}(x_0, y_1(x_0)) \\ \wedge & \mathsf{Loves}(y_0, x_1(y_0)) \end{array} \right) \Rightarrow \left(\begin{array}{ll} & \mathsf{Loves}(y_1(x_0), x_1(y_0)) \\ \wedge & \mathsf{Loves}(x_1(y_0), y_1(x_0)) \end{array} \right) \right) \qquad (25\mathrm{b}')$$

Indeed, it is easy to see from (H2) that the polarity of the first two (negative) occurrences of the Loves-predicate in (25b′) must actually be positive.

Note 11 (Technical Disadvantages of Heusinger's Indexed ε-operator)

When trying to understand the semantics of sentences in natural language, it might be the case that a representation of the indefinite article with (a variant of) our new indefinite semantics for the ε offers the following advantages compared to [Heusinger, 1997]:

1. We do not have to disambiguate a specific from a non-specific usage in advance, contrary to [Heusinger, 1997] where we have to choose between $G(\varepsilon_l x.\, F(x))$ and $\exists i. G(\varepsilon_i x.\, F(x))$ eagerly. Besides this, the design decision to pack the information on specificity into the ε-term may be questioned.

2. For a computer implementation, the l and i in these formulas have to be implemented as something isomorphic to free δ^+-variables (or free γ-variables) anyway, so that our representation (i.e. $G(x^{\delta^+})$ with choice-condition $C(x^{\delta^+}) := F(x^{\delta^+})$) saves one level of indirection.

3. Our possibility of a formally verified instantiation of free δ^+-variables (cf. § 4.6 and Theorem 5.21(6)) could provide a formal means in the stepwise process of approaching the intended semantics of sentences in natural language.

Note 12 The well-foundedness required for the soundness of *descente infinie* gave rise to a notion of reduction which preserves solutions, cf. Definition 5.20. The liberalized δ-rules as found in [Fitting, 1996] do not satisfy this notion. The addition of our choice-conditions finally turned out to be the only way to repair this defect of the liberalized δ-rules. Cf. [Wirth, 2004] for more details.

References

[Ackermann, 1938] Wilhelm Ackermann. Mengentheoretische Begründung der Logik. *Mathematische Annalen*, 115:1–22, 1938. Received April 23, 1937.

[Ackermann, 1940] Wilhelm Ackermann. Zur Widerspruchsfreiheit der Zahlentheorie. *Mathematische Annalen*, 117:163–194, 1940. Received Aug. 15, 1939.

[Andrews, 2002] Peter B. Andrews. *An Introduction to Mathematical Logic and Type Theory: To Truth Through Proof.* Academic Press (Elsevier), 2002. 2nd ed. (1st ed. 1986).

[Anon, 1905] Anon, editor. *Verhandlungen des Dritten Internationalen Mathematiker-Kongresses, Heidelberg, Aug. 8–13, 1904.* Verlag von B. G. Teubner, Leipzig, 1905.

[Asser, 1957] Günter Asser. Theorie der logischen Auswahlfunktionen. *Zeitschrift für math. Logik und Grundlagen der Math.*, 3:30–68, 1957.

[Asser, 1990] Günter Asser, editor. *Guiseppe Peano — Arbeiten zur Analysis und zur mathematischen Logik.* Number 13 in Teubner-Archiv zur Mathematik. B. G. Teubner Verlagsgesellschaft, 1990.

[Bell &al., 2001] John Lane Bell, David DeVidi, and Graham Solomon. *Logical Options: An Introduction to Classical and Alternative Logics.* Broadview Press, 2001.

[Bell, 1993a] John Lane Bell. Hilbert's ε-Operator and Classical Logic. *J. Philosophical Logic*, 22:1–18, 1993.

[Bell, 1993b] John Lane Bell. Hilbert's ε-Operator in Intuitionistic Type Theories. *Math. Logic Quart.*, 39:323–337, 1993.

[Berka & Kreiser, 1973] Karel Berka and Lothar Kreiser, editors. *Logik-Texte – Kommentierte Auswahl zur Geschichte der modernen Logik.* Akademie-Verlag, Berlin, 1973. 2nd rev. ed. (1st ed. 1971; 4th rev. rev. ed. 1986).

[Bibel & Schmitt, 1998] Wolfgang Bibel and Peter H. Schmitt, editors. *Automated Deduction — A Basis for Applications.* Kluwer (Springer), 1998.

[Blass & Gurevich, 2000] Andreas Blass and Yuri Gurevich. The logic of choice. *J. Symbolic Logic*, 65:1264–1310, 2000.

[Bourbaki, 1939ff.] Nicolas Bourbaki. *Théorie des Ensembles.* Éléments des Mathématique. Hermann, Paris, 1939ff..

[Caferra & Salzer, 2000] Ricardo Caferra and Gernot Salzer, editors. *Automated Deduction in Classical and Non-Classical Logics.* Number 1761 in Lecture Notes in Artificial Intelligence. Springer, 2000.

[Costa &al., 1991] Newton C. A. da Costa, F. A. Doria, and N. Papavero. Meinong's Theory of Objects and Hilbert's ε-symbol. *Reports on Mathematical Logic*, 25:119–132, 1991. ISSN 0137–2904, Jagiellonian University, Krakow.

[Deemter & Peters, 1996] Kees van Deemter and Stanley Peters, editors. *Semantic Ambiguity and Underspecification*. CLSI LN No. 55, Stanford, 1996.

[DeVidi, 1995] David DeVidi. Intuitionistic ε- and τ-calculi. *Math. Logic Quart.*, 41:523–546, 1995.

[Eijck, 1993] Jan van Eijck. The dynamics of description. *Journal of Semantics*, 10:239–267, 1993.

[Fitting, 1975] Melvin Fitting. A modal logic ε-calculus. *Notre Dame J. of Formal Logic*, XVI:1–16, 1975.

[Fitting, 1996] Melvin Fitting. *First-order logic and automated theorem proving*. Springer, 1996. 2nd rev. ed. (1st ed. 1990).

[Fitting, 1999] Melvin Fitting. On quantified modal logic. *Fundamenta Informaticae*, 39:105–121, 1999.

[Frege, 1879] Gottlob Frege. *Begriffsschrift, eine der arithmetischen nachgebildete Formelsprache des reinen Denkens*. Verlag von L. Nebert, Halle an der Saale, 1879. Corrected facsimile in [Frege, 1964]. Reprint of pp. III–VIII and pp. 1–54 in [Berka & Kreiser, 1973, pp. 48–106]. English translation in [Heijenoort, 1971, pp. 1–82].

[Frege, 1893/1903] Gottlob Frege. *Grundgesetze der Arithmetik – Begriffsschriftlich abgeleitet*. Verlag von Hermann Pohle, Jena, 1893/1903. As facsimile with corrigenda by Christian Thiel: Georg Olms Verlag, Hildesheim, 1998.

[Frege, 1964] Gottlob Frege. *Begriffsschrift und andere Aufsätze*. Wissenschaftliche Buchgesellschaft, Darmstadt, 1964. Zweite Auflage, mit Edmund Husserls und Heinrich Scholz' Anmerkungen, herausgegeben von Ignacio Angelelli.

[Gabbay & Guenthner, 2001ff.] Dov Gabbay and F. Guenthner, editors. *Handbook of Philosophical Logic, 2nd ed.*. Springer, 2001ff..

[Gabbay, 1981] Dov Gabbay. *Semantical Investigations in Heyting's Intuitionistic Logic*. Kluwer (Springer), 1981.

[Gentzen, 1935] Gerhard Gentzen. Untersuchungen über das logische Schließen. *Mathematische Zeitschrift*, 39:176–210,405–431, 1935. Also in [Berka & Kreiser, 1973, pp. 192–253]. English translation in [Gentzen, 1969].

[Gentzen, 1969] Gerhard Gentzen. *The Collected Papers of Gerhard Gentzen*. North-Holland (Elsevier), 1969. Ed. by Manfred E. Szabo.

[Geurts, 2000] Bart Geurts. Indefiniteness and choice functions. *Linguistic Inquiry*, 31:731–738, 2000.

[Giese & Ahrendt, 1999] Martin Giese and Wolfgang Ahrendt. Hilbert's ε-terms in automated theorem proving. In Neil Murray, editor, *8th Int. Conf. on Tableaus and Related Methods, Saratoga Springs (NY), 1999*, number 1617 in Lecture Notes in Artificial Intelligence, pages 171–185. Springer, 1999.

[Gillman, 1987] Leonard Gillman. *Writing Mathematics Well.* The Mathematical Association of America, 1987.

[Heijenoort, 1971] Jean van Heijenoort. *From Frege to Gödel: A Source Book in Mathematical Logic, 1879–1931.* Harvard Univ. Press, 1971. 2nd rev. ed. (1st ed. 1967).

[Hermes, 1965] Hans Hermes. *Eine Termlogik mit Auswahloperator.* Number 6 in LNM. Springer, 1965.

[Heusinger & Egli, 1996] Klaus von Heusinger and Urs Egli, editors. *Proc. of the Konstanz Workshop "Reference and Anaphorical Relations"*, number 79, 1996.

[Heusinger, 1996] Klaus von Heusinger. The reference of indefinites. 1996. In [Heusinger & Egli, 1996, pp. 73–92]. `http://www.ub.uni-konstanz.de/kops/volltexte/2000/508/pdf/ap079_4.pdf`.

[Heusinger, 1997] Klaus von Heusinger. *Salienz und Referenz — Der Epsilonoperator in der Semantik der Nominalphrase und anaphorischer Pronomen.* Number 43 in Studia grammatica. Akademie Verlag, Berlin, 1997.

[Hilbert & Bernays, 1968/70] David Hilbert and Paul Bernays. *Die Grundlagen der Mathematik.* Springer, 1968/70. 2nd rev. ed. (1st ed. 1934/39).

[Hilbert, 1899] David Hilbert. *Grundlagen der Geometrie.* Verlag von B. G. Teubner, Leipzig, 1899. In *Festschrift zur Feier der Enthüllung des Gauß-Weber-Denkmals in Göttingen, herausgegeben von dem Fest-Comitee.* 1st ed. without appendixes. Last edition by Hilbert is [Hilbert, 1930a], which is also most complete regarding the appendixes. Last three editions by Paul Bernays are [Hilbert, 1962; 1968; 1972], which are also most complete regarding supplements and figures. Two substantially different English translations are [Hilbert, 1902] and [Hilbert, 1971].

[Hilbert, 1900] David Hilbert. Über den Zahlbegriff. *Jahresbericht der Deutschen Mathematiker-Vereinigung*, 8:180–184, 1900. Reprinted as Appendix VI of [Hilbert, 1909; 1913; 1922; 1923b; 1930a].

[Hilbert, 1902] David Hilbert. *The Foundations of Geometry.* Open Court, Chicago, 1902. Authorized English translation of [Hilbert, 1899] by E. J. Townsend, `http://www.gutenberg.org/etext/17384`.

[Hilbert, 1903] David Hilbert. *Grundlagen der Geometrie. Zweite, durch Zusätze vermehrte und mit fünf Anhängen versehene Auflage. Mit zahlreichen in den Text gedruckten Figuren.* Druck und Verlag von B. G. Teubner, Leipzig, 1903. 2nd rev. extd. ed. of [Hilbert, 1899], rev. and extd. with five appendixes, newly added figures, and an index of notion names.

[Hilbert, 1905] David Hilbert. Über die Grundlagen der Logik und der Arithmetik. 1905. In [Anon, 1905, 174–185]. Reprinted as Appendix VII of [Hilbert, 1909; 1913; 1922; 1923b; 1930a]. English translation *On the foundations of logic and arithmetic* by Beverly Woodward with an introduction by Jean van Heijenoort in [Heijenoort, 1971, pp. 129–138].

[Hilbert, 1909] David Hilbert. *Grundlagen der Geometrie. Dritte, durch Zusätze und Literaturhinweise von neuem vermehrte und mit sieben Anhängen versehene Auflage. Mit zahlreichen in den Text gedruckten Figuren.* Druck und Verlag von B. G. Teubner, Leipzig, Berlin, 1909. 3rd rev. extd. ed. of [Hilbert, 1899], rev. ed. of [Hilbert, 1903], extd. with a bibliography and two additional appendixes (now seven in total) (Appendix VI: [Hilbert, 1900]) (Appendix VII: [Hilbert, 1905]).

[Hilbert, 1913] David Hilbert. *Grundlagen der Geometrie. Vierte, durch Zusätze und Literaturhinweise von neuem vermehrte und mit sieben Anhängen versehene Auflage. Mit zahlreichen in den Text gedruckten Figuren.* Druck und Verlag von B. G. Teubner, Leipzig, Berlin, 1913. 4th rev. extd. ed. of [Hilbert, 1899], rev. ed. of [Hilbert, 1909].

[Hilbert, 1922] David Hilbert. *Grundlagen der Geometrie. Fünfte, durch Zusätze und Literaturhinweise von neuem vermehrte und mit sieben Anhängen versehene Auflage. Mit zahlreichen in den Text gedruckten Figuren.* Verlag und Druck von B. G. Teubner, Leipzig, Berlin, 1922. 5th extd. ed. of [Hilbert, 1899]. Contrary to what the sub-title says, this is an anastatic reprint of [Hilbert, 1913], extended with a very short preface on the changes w.r.t. [Hilbert, 1913], and with augmentations to Appendix II, Appendix III, and Chapter IV, § 21.

[Hilbert, 1923a] David Hilbert. Die logischen Grundlagen der Mathematik. *Mathematische Annalen*, 88:151–165, 1923.

[Hilbert, 1923b] David Hilbert. *Grundlagen der Geometrie. Sechste unveränderte Auflage. Anastatischer Nachdruck. Mit zahlreichen in den Text gedruckten Figuren.* Verlag und Druck von B. G. Teubner, Leipzig, Berlin, 1923. 6th rev. extd. ed. of [Hilbert, 1899], anastatic reprint of [Hilbert, 1922].

[Hilbert, 1926] David Hilbert. Über das Unendliche. *Mathematische Annalen*, 88:151–165, 1926. Received June 24, 1925. Reprinted as Appendix VIII of [Hilbert, 1930a]. English translation *On the infinite* by Stefan Bauer-Mengelberg with an introduction by Jean van Heijenoort in [Heijenoort, 1971, pp. 367–392].

[Hilbert, 1928] David Hilbert. Die Grundlagen der Mathematik – Vortrag, gehalten auf Einladung des Mathematischen Seminars im Juli 1927 in Hamburg. *Abhandlungen aus dem mathematischen Seminar der Univ. Hamburg*, 6:65–85, 1928. Reprinted as Appendix IX of [Hilbert, 1930a]. English translation *The foundations of mathematics* by Stefan Bauer-Mengelberg and Dagfinn Føllesdal with a short introduction by Jean van Heijenoort in [Heijenoort, 1971, pp. 464–479].

[Hilbert, 1930a] David Hilbert. *Grundlagen der Geometrie. Siebente umgearbeitete und vermehrte Auflage. Mit 100 in den Text gedruckten Figuren.* Verlag und Druck von B. G. Teubner, Leipzig, Berlin, 1930. 7th rev. extd. ed. of [Hilbert, 1899], thoroughly revised edition of [Hilbert, 1923b], extd. with three new appendixes (now ten in total) (Appendix VIII: [Hilbert, 1926]) (Appendix IX: [Hilbert, 1928]) (Appendix X: [Hilbert, 1930b]).

[Hilbert, 1930b] David Hilbert. Probleme der Grundlegung der Mathematik. *Mathematische Annalen*, 102:1–9, 1930. Vortrag gehalten auf dem Internationalen Mathematiker-Kongreß in Bologna, Sept. 3, 1928. Received March 25, 1929. Reprinted as Appendix X

of [Hilbert, 1930a]. Short version in *Atti del congresso internationale dei matematici, Bologna, 3–10 settembre 1928*, Vol. 1, pp. 135–141, Bologna, 1929.

[Hilbert, 1956] David Hilbert. *Grundlagen der Geometrie. Achte Auflage, mit Revisionen und Ergänzungen von Dr. Paul Bernays. Mit 124 Abbildungen.* B. G. Teubner Verlagsgesellschaft, Stuttgart, 1956. 8th rev. extd. ed. of [Hilbert, 1899], rev. ed. of [Hilbert, 1930a], omitting appendixes VI–X, extd. by Paul Bernays, now with 24 additional figures and 3 additional supplements.

[Hilbert, 1962] David Hilbert. *Grundlagen der Geometrie. Neunte Auflage, revidiert und ergänzt von Dr. Paul Bernays. Mit 129 Abbildungen.* B. G. Teubner Verlagsgesellschaft, Stuttgart, 1962. 9th rev. extd. ed. of [Hilbert, 1899], rev. ed. of [Hilbert, 1956], extd. by Paul Bernays, now with 129 figures, 5 appendixes, and 8 supplements (I 1, I 2, II, III, IV 1, IV 2, V 1, V 2).

[Hilbert, 1968] David Hilbert. *Grundlagen der Geometrie. Zehnte Auflage, revidiert und ergänzt von Dr. Paul Bernays. Mit 124 Abbildungen.* B. G. Teubner Verlagsgesellschaft, Stuttgart, 1968. 10th rev. extd. ed. of [Hilbert, 1899], rev. ed. of [Hilbert, 1962] by Paul Bernays.

[Hilbert, 1971] David Hilbert. *The Foundations of Geometry.* Open Court, Chicago and La Salle (IL), 1971. Newly translated and fundamentally different 2nd ed. of [Hilbert, 1902], actually an English translation of [Hilbert, 1968] by Leo Unger.

[Hilbert, 1972] David Hilbert. *Grundlagen der Geometrie. 11. Auflage. Mit Supplementen von Dr. Paul Bernays.* B. G. Teubner Verlagsgesellschaft, Stuttgart, 1972. 11th rev. extd. ed. of [Hilbert, 1899], rev. ed. of [Hilbert, 1968] by Paul Bernays.

[Hintikka, 1974] K. Jaakko J. Hintikka. Quantifiers vs. quantification theory. *Linguistic Inquiry*, V(2):153–177, 1974.

[Hintikka, 1996] K. Jaakko J. Hintikka. *The Principles of Mathematics Revisited.* Cambridge Univ. Press, 1996.

[Hobbs & Schieber, 1987] Jerry R. Hobbs and Stuart M. Schieber. An algorithm for generating quantifier scopings. *Computational Linguistics*, 13:47–63, 1987.

[Hobbs, 1996] Jerry R. Hobbs. Monotone decreasing quantifiers in a scope-free logical form. 1996. In [Deemter & Peters, 1996, pp. 55–76]. `http://www.isi.edu/~hobbs/monotone-decreasing.pdf`.

[Hobbs, 2003ff.] Jerry R. Hobbs. Discourse and inference. Magnum Opus in Progress, Web only. `http://www.isi.edu/~hobbs/disinf-tc.html`, 2003ff..

[Howard & Rubin, 1998] Paul Howard and Jean E. Rubin. *Consequences of the Axiom of Choice.* American Math. Society, 1998.

[Kamp & Reyle, 1993] Hans Kamp and Uwe Reyle. *From Discourse to Logic.* Kluwer (Springer), 1993.

[Kamp &al., 2005] Hans Kamp, Josef van Genabith, and Uwe Reyle. Discourse representation theory. 2005. In [Gabbay & Guenthner, 2001ff.,]. http://www.ims.uni-stuttgart.de/~uwe/Papers/DRT.pdf.

[Kennedy, 1973] Hubert C. Kennedy. *Selected works of Guiseppe Peano.* George Allen & Unwin, London, 1973.

[Kennedy, 2002] Hubert C. Kennedy. *Peano — Life and Works of Guiseppe Peano.* 2002. Corrected ed. (1st ed. 1980, D. Reidel Publ., Dordrecht), Peremptory Publications ebook, http://home.att.net/~clairnorman/Peano2002.pdf.

[Kühler & Wirth, 1996] Ulrich Kühler and Claus-Peter Wirth. *Conditional Equational Specifications of Data Types with Partial Operations for Inductive Theorem Proving.* SEKI-Report SR–1996–11 (ISSN 1437–4447). SEKI Publications, Univ. Kaiserslautern, 1996. Short version in 8th RTA 1997, LNCS 1232, pp. 38–52, Springer. http://www.ags.uni-sb.de/~cp/p/rta97.

[Kohlhase, 1998] Michaël Kohlhase. Higher-order automated theorem proving. 1998. In [Bibel & Schmitt, 1998, Vol. 1, pp. 431–462].

[Koller, 2004] Alexander Koller. *Constraint-Based and Graph-Based Resolution of Ambiguities in Natural Language.* PhD thesis, Universität des Saarlandes, 2004.

[Leisenring, 1969] Albert C.Leisenring. *Mathematical Logic and Hilbert's ε-Symbol.* Gordon and Breach, New York, 1969.

[Mann, 1898] Thomas Mann. *Der kleine Herr Friedemann.* S. Fischer Verlag, Berlin, 1898.

[Mattick & Wirth, 1999] Volker Mattick and Claus-Peter Wirth. An algebraic Dexter-based hypertext reference model. Research Report (green/grey series) 719/1999, FB Informatik, Univ. Dortmund, 1999. http://www.ags.uni-sb.de/~cp/p/gr719, http://arxiv.org/abs/0902.3648.

[Meinong, 1904a] Alexius Meinong. Über Gegenstandstheorie. 1904. In [Meinong, 1904b, pp. 1–50].

[Meinong, 1904b] Alexius Meinong, editor. *Untersuchungen zur Gegenstandstheorie und Psychologie.* Barth, Leipzig, 1904.

[Meyer-Viol, 1995] Wilfried P. M. Meyer-Viol. *Instantial Logic — An Investigation into Reasoning with Instances.* PhD thesis, Univ. Utrecht, 1995. ILLC dissertation series 1995–11.

[Miller, 1992] Dale A. Miller. Unification under a mixed prefix. *J. Symbolic Computation*, 14:321–358, 1992.

[Nipkow &al., 2000] Tobias Nipkow, Lawrence C. Paulson, and Markus Wenzel. ISABELLE's Logics: HOL. Web only: http://isabelle.in.tum.de/PSV2000/doc/logics-HOL.pdf, 2000.

[Nipkow &al., 2002] Tobias Nipkow, Lawrence C. Paulson, and Markus Wenzel. ISABELLE/HOL — *A Proof Assistant for Higher-Order Logic*. Number 2283 in Lecture Notes in Computer Science. Springer, 2002.

[Parsons, 1994] Terence Parsons. Anaphoric pronouns in very late medieval supposition theory. *Linguistics and Philosophy*, 17:429–445, 1994. `http://www.humnet.ucla.edu/humnet/phil/faculty/tparsons/PRONOUNS.doc`.

[Peano, 1884] Guiseppe Peano, editor. *Angelo Genocchi — Calcolo differenziale e principii di calcolo integrale*. Fratelli Bocca, Torino, 1884. German translation: [Peano, 1899a].

[Peano, 1890] Guiseppe Peano. Démonstration de l'intégrabilité des équations différentielles ordinaires. *Mathematische Annalen*, 37:182–228, 1890. Facsimile also in [Asser, 1990, pp. 76–122].

[Peano, 1896f.] Guiseppe Peano. Studii di logica matematica. *Atti della Reale Accademia delle Scienze di Torino — Classe di Scienze Morali, Storiche e Filologiche e Classe di Scienze Fisiche, Matematiche e Naturali*, 32:565–583, 1896f.. Also in Atti della Reale Accademia delle Scienze di Torino — Classe di Scienze Fisiche, Matematiche e Naturali **32**, pp. 361–397. English translation *Studies in Mathematical Logic* in [Kennedy, 1973, pp. 190–205]. German translation: [Peano, 1899b].

[Peano, 1899a] Guiseppe Peano, editor. *Angelo Genocchi — Differentialrechnung und Grundzüge der Integralrechnung*. B. G. Teubner Verlagsgesellschaft, Leipzig, 1899. German translation of [Peano, 1884].

[Peano, 1899b] Guiseppe Peano. Über mathematische Logik. 1899. German translation of [Peano, 1896f.]. In [Peano, 1899a, Appendix 1]. Facsimile also in [Asser, 1990, pp. 10–26].

[Peirce, 1885] Charles S. Peirce. On the algebra of logic: A contribution to the philosophy of notation. *American J. of Mathematics*, 7:180–202, 1885. Also in [Peirce, 1993, pp. 162–190].

[Peirce, 1993] Charles S. Peirce. *Writings of Charles S. Peirce — A Chronological Edition, Vol. 5, 1884–1886*. Indiana Univ. Press, 1993. Ed. by Christian J. W. Kloesel.

[Pinkal &al., 2001] Manfred Pinkal, Jörg Siekmann, and Christoph Benzmüller. Teilprojekt MI 3: DIALOG: Tutorieller Dialog mit einem Mathematik-Assistenten. 2001. In SFB 378 Resource-Adaptive Cognitive Processes, Proposal Jan. 2002 - Dez. 2004, Saarland Univ..

[Prawitz, 1960] Dag Prawitz. An improved proof procedure. 1960. In [Siekmann & Wrightson, 1983, Vol. 1, pp. 159–199].

[Prawitz, 1965] Dag Prawitz. *Natural Deduction*. Almquist & Wiksells, Uppsala, 1965.

[Priest & Tanaka, 2009] Graham Priest and Koji Tanaka. Paraconsistent logic. In Edward N. Zalta, editor, *The Stanford Encyclopedia of Philosophy*. Stanford University, Web only, 2009. Ed. Spring 2009, `http://plato.stanford.edu/archives/spr2009/entries/logic-paraconsistent`.

[Quine, 1981] Willard Van O. Quine. *Mathematical Logic*. Harvard Univ. Press, 1981. 4th rev. ed. (1st ed. 1940).

[Rubin & Rubin, 1985] Herman Rubin and Jean E. Rubin. *Equivalents of the Axiom of Choice*. North-Holland (Elsevier), 1985. 2nd rev. ed. (1st ed. 1963).

[Russell, 1905a] Bertrand Russell. On denoting. *Mind*, 14:479–493, 1905.

[Russell, 1905b] Bertrand Russell. Review of [Meinong, 1904b]. *Mind*, 14:537–537, 1905.

[Russell, 1919] Bertrand Russell. *Introduction to Mathematical Philosophy*. George Allen & Unwin, London, 1919.

[Schmidt-Samoa, 2006] Tobias Schmidt-Samoa. An even closer integration of linear arithmetic into inductive theorem proving. *Electronic Notes in Theoretical Computer Sci.*, 151:3–20, 2006. `http://www.ags.uni-sb.de/~cp/p/evencloser`, `http://dx.doi.org/10.1016/j.entcs.2005.11.020`.

[Siekmann & Wrightson, 1983] Jörg Siekmann and Graham Wrightson, editors. *Automation of Reasoning*. Springer, 1983.

[Smullyan, 1968] Raymond M. Smullyan *First-Order Logic*. Springer, 1968.

[Wallen, 1990] Lincoln A. Wallen. *Automated Proof Search in Non-Classical Logics*. MIT Press, 1990.

[Whitehead & Russell, 1910–1913] Alfred North Whitehead and Bertrand Russell. *Principia Mathematica*. Cambridge Univ. Press, 1910–1913. 1st ed..

[Wirth, 1998] Claus-Peter Wirth. Full first-order sequent and tableau calculi with preservation of solutions and the liberalized δ-rule but without Skolemization. Research Report (green/grey series) 698/1998, FB Informatik, Univ. Dortmund, 1998. Short version in Gernot Salzer, Ricardo Caferra (eds.). Proc. 2nd Int. Workshop on First-Order Theorem Proving (FTP'98), pp. 244–255, Tech. Univ. Vienna, 1998. Short version also in [Caferra & Salzer, 2000, pp. 283–298]. `http://www.ags.uni-sb.de/~cp/p/ftp98`, `http://arxiv.org/abs/0902.3730`.

[Wirth, 2002] Claus-Peter Wirth. A new indefinite semantics for Hilbert's epsilon. In Uwe Egly and Christian G. Fermüller, editors, *11th Int. Conf. on Tableaus and Related Methods, København, 2002*, number 2381 in Lecture Notes in Artificial Intelligence, pages 298–314. Springer, 2002. `http://www.ags.uni-sb.de/~cp/p/epsi`.

[Wirth, 2004] Claus-Peter Wirth. Descente Infinie + Deduction. *Logic J. of the IGPL*, 12:1–96, 2004. `http://www.ags.uni-sb.de/~cp/p/d`.

[Wirth, 2006] Claus-Peter Wirth. lim+, *δ^+, and Non-Permutability of β-Steps*. SEKI-Report SR–2005–01 (ISSN 1437–4447). SEKI Publications, Saarland Univ., 2006. Rev. ed. `http://arxiv.org/abs/0902.3635`.

[Wirth, 2009] Claus-Peter Wirth. Shallow confluence of conditional term rewriting systems. *J. Symbolic Computation*, 44:69–98, 2009. `http://dx.doi.org/10.1016/j.jsc.2008.05.005`.